Co-funded by the European Union

PROJECT NUMBER: 2022-1-DE-03-KA2020-SCH-000087039

STEMSiL

STEMSiL Handbook

Research in STEM Teaching and Learning in Sign Languages

Herausgeberinnen und Herausgeber:

Unterhitzenberger, G., Nordheimer, S., Peters, C., Schmidt, F., Schuler, L., Bumann, S. & Rathmann, C.

Contents:

1. From Scientific Concepts to Signs – Leveraging STEM Glossaries in Deaf Education

Audrey M. Cameron

This chapter explores the innovative use of STEM sign language glossaries in deaf education, focusing on the Scottish Sensory Centre (SSC) BSL Glossary project. It examines the critical intersection of deaf education, sign language, and STEM subjects, highlighting the challenges and opportunities in teaching complex scientific concepts to deaf students. The chapter traces the evolution of sign language glossaries and details the meticulous process of developing scientific signs. Through examples from various scientific disciplines, including geography, biology, astronomy, and chemistry, it demonstrates how carefully crafted signs can bridge the gap between written scientific terminology and visual-spatial cognition. The chapter also discusses the impact of these resources on conceptual understanding, presenting evidence from classroom observations and research. By exploring the development and application of sign language in STEM education, this work illuminates the transformative potential of visual language in conveying complex scientific ideas, enhancing accessibility, and promoting equal opportunities for deaf learners in STEM fields.

This is a Transcript of the IS videos

The author gratefully acknowledges the contributions of the SSC BSL Glossary team, including Gary Quinn and Rachel O'Neill, whose collaborative efforts in developing the signs and concepts discussed in this chapter were invaluable. The IS videos were translated by David Summersgill (sign language interpreter), and the images were produced by Abigail Sheridan and Molly McInulty.

1.1 Introduction: Bridging STEM and Sign Language

https://edin.ac/4fc2fLl

This chapter explores the critical intersection of deaf education, sign language, and STEM subjects, focusing on the work of the Scottish Sensory Centre (SSC) BSL Glossary project. As we delve into the challenges and opportunities in teaching complex scientific concepts to deaf students, we will examine the evolution of sign language glossaries, the meticulous process of developing scientific signs, and the impact of these resources on conceptual understanding. From the visualisation of geographical landscapes to the representation of abstract chemical processes, this chapter illuminates how carefully crafted signs can bridge the gap between written scientific terminology and visual-spatial cognition. By exploring examples across various scientific disciplines, we aim to demonstrate the transformative potential of sign language in STEM education for deaf learners.

1.2. Teaching Science and STEM Through Conceptual Understanding

https://edin.ac/3Ls4Xin

Driver et al. (2014) found that providing learners with opportunities to conduct experiments, engage in 'hands-on' activities and dialogue with their peers is essential. These experiences are crucial for facilitating understanding. More successful learners typically grow up in environments where they can understand those around them and use that knowledge to make connections when they start school, which is critical for their development. Deaf students, however, require more examples to give them the necessary understanding to grasp concepts (Jones 2014; Flores & Rumjanek, 2015; Cameron et al., 2017). Driver et al. (1994, 2014) emphasise that children need access to dialogue to interpret and understand experiments and activities collaboratively. Teachers play a vital role in this process by guiding students and contributing to their construction of meaning. This cannot be done solely by the students themselves. Teachers should ask probing questions like 'Why?' to assess students' understanding and to stimulate their critical thinking.

Deaf people often have fewer opportunities to participate in these experiences and environments, most effective when all parties can communicate fluently in sign language. This allows for the development of an understanding of scientific concepts (Lindahl, 2015; Mercer & Littleton, 2007). We all construct our way of looking at the world, and it is important to ask children about their perspectives, which may differ from our own. By understanding a pupil's 'worldview,' educators can transform their thinking through teaching. Students need opportunities to explore the world outside the classroom. Reading about concepts isn't enough; they require access to learning in diverse learning modalities, such as pictures, experiments, outdoor activities, and films (Jones, 2014; Raven & Whitman, 2019; Lindahl, 2015 & 2021). Deaf children need the entire learning experience, with a strong emphasis on dialogue. This comprehensive approach ensures they develop a complete grasp of concepts.

1.3. History and Evolution of Sign Language Glossaries

https://edin.ac/3ScSF1c

Sign language glossaries/lexicons have existed for many years (McKee & Vale, 2017). Initially, signs were recorded in books using drawings of the signs accompanied by written words. Then came photos – still images, often arranged in sequence to show the movements of the signs. Some images included notations to indicate the handshape and movements, sometimes with arrows to show direction (Brien, 1993).

With the advent of film, the movement of signs could be captured more fully. VHS and video recorders were later followed by CD-ROM(Signs for Education – the definite BSL reference for education), DVDs. Now, the internet allows signs to be filmed and uploaded to websites which can be seen globally (Scottish Sensory Centre's STEM in BSL glossary: https://www.ssc.education.ed.ac.uk/BSL/environment/interdependence.html). In the past, signs in books were static, but today, video clips can be easily replaced and updated. The web has also enabled glossaries to grow in size.

Lang was the first to set up a website of signs linked to STEM at the National Technical Institute for the Deaf at Rochester Institute of Technology (NTID/RIT) (Lang et al., 2007). Since then, glossaries have grown significantly. In 2023/24, the Global Year of STEM Sign Language Lexicons brought together various groups developing glossaries worldwide to share their work (Global Year of STEM Sign Language Lexicons 2023-2024). This gathering allowed us to meet, support one another, and discuss our strategies.Different groups create their glossaries in various ways (Cohen, 2024). Some use a 'self-load' method, inviting individuals to film and post signs for specific terms, thereby creating a corpus of signs. Other groups follow a more collaborative approach, where signs are identified, recommended or developed through group discussions (SSC signs development project: https://www.ssc.education.ed.ac.uk/BSL/index.html#top) (Cameron et al., 2019; O'Neill et al., 2020). Some glossaries have been compiled by sign language interpreters or educators. Uploading video files to the web has made this process much more manageable. Look at the list of different glossaries from around the world, all aiming to facilitate better access to STEM for deaf people (Table 1).

1.4. Purpose of SSC BSL Glossary

https://edin.ac/46cz0nI

The Scottish Sensory Centre (SSC)'s BSL Glossary was established following research conducted by Dr Mary Brennan in 2000. At that time, she researched sign linguistics at the University of Edinburgh and investigated the challenges deaf pupils faced accessing national examinations. Mary subsequently wrote to the Scottish Qualifications Authority (SQA) to request fair access for deaf students, proposing that teachers be allowed to sign the exam questions and that deaf students give their answers in sign language (Brennan, 2000). The SQA approved this in 2000, but then Mary quickly identified a significant issue – a shortage of signs for the STEM vocabulary.

In response, Mary collaborated with Gerry Hughes, a deaf maths teacher, and together, they created a pilot glossary for Maths, which initially contained 90 signs (https://www.ssc.education.ed.ac.uk/BSL/maths.html). The response to the new glossary was positive and demonstrated the need for a more comprehensive one. Since then, with intermittent funding, the SSC glossary has continued to expand. Further demonstrating the demand for a glossary, research commissioned by the Royal Society in 2018 surveyed the number of disabled students pursuing STEM subjects in Higher Education in the UK over ten years from 2007/8 to 2018/9 (Joice & Tetlow, 2021). For deaf students, the figure was only 0.3% in 2008, and 10 years later, it was found that whilst the number of disabled students had increased overall, the percentage of deaf students remained at 0.3%.

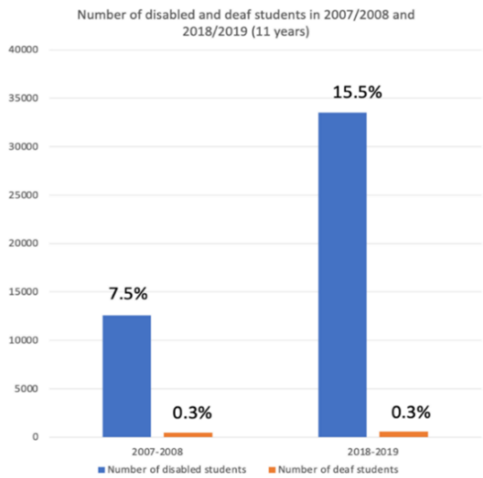


Figure 1: Graph showing the number of disabled students studying STEM in Higher Education over ten years (2007/8 to 2018/19). (Joice & Tetlow, 2021).

This stagnation persisted despite advancements in support, such as the Disabled Students' Allowance (DSA) and access to sign language interpreters and notetakers. The primary cause seemed to be the limited access to STEM sign vocabulary for deaf students at school and university. Educators and sign language interpreters need access to a glossary of STEM signs to support deaf students better (Cameron et al., 2017).

In the United States, Lualdi et al. (2023) wrote an article about deaf people's experiences of learning and working in the field of STEM. This article also identified a lack of access to appropriate signs for STEM terms as one of the barriers. This barrier also included interpreters not having the signs, which meant it was hard for deaf students to discuss their work with fellow students and colleagues. These experiences demonstrate the importance of sign glossaries.

1.5. Structure and Content of the SSC BSL Glossary

https://edin.ac/3yesfFc

The SSC Glossary Website homepage (https://www.ssc.education.ed.ac.uk/BSL/index.html#top) features a number of tiles (pictures/graphics), each representing different topics such as Astronomy, Biology, Chemistry and Geography. Clicking on the tile will expand to show (https://www.ssc.education.ed.ac.uk/BSL/environmenthome.html):

- Left Side: A list of subtopics
- Right Side: An A-Z menu for alphabetical browsing.

Alphabetical List This A-Z list is useful if you encounter a scientific term and need the equivalent sign. For example, to find the scientific term starting with 'B', click on 'B' and select from the dropdown menu.

Topic-Based Signs. On the other side of the screen, signs are grouped according to topic within the subject area (https://www.ssc.education.ed.ac.uk/BSL/environment/theme1.html). Clicking on a tile brings up terms related to that topic along with the corresponding signs, which can be helpful for:

- Teachers teaching a specific topic
- Interpreters learning relevant signs of a specific topic
- Students learn from home and can watch the explanation that goes with the signs.

Video Demonstrations Clicking on any scientific terms brings up a video of the sign. Below the video is an option for 'explanation' or 'definition', which, when clicked, shows a signed explanation of the term along with a written English translation. This makes the Glossary a bilingual resource, with signs for terms, signed definitions and English translations. The signed definitions are not translations from written texts in textbooks; rather, the text is a translation of the signed videos (O'Neill et al., 2019).

Example videos In 2008, we asked students what they thought of the glossary when they were looking at it; they told us they wanted more video examples so that they could see how the signs related to science in the lab or outside (for example, actual topography), i.e., real-life examples. So, we've added another link under the term that, when clicked, brings up these example videos (Cameron et al, 2012, 2017).

A short reel showing different examples from the SSC Glossary (Table 2).

- Distillation ,
- Corrie ,
- Stamen ,
- Mixture and Separation ,
- Reflecting Telescope ,

Feedback and Use Feedback from users (students, teachers and interpreters) has been positive, highlighting that the signs, explanations, and examples enhanced understanding of scientific concepts for young people (Cameron et al., 2017). Teachers have said the Glossary has helped them teach and how to explain the concepts through sign language. The same has been true for interpreters working in schools and universities, who have been uncertain about signing STEM content. Teachers have also been using the signs in the SSC glossary in their lessons because they found they help their non-deaf pupils' to understand complex scientific concepts (Hickman, 2013). On-screen presenters also provided interpretations using signs from the glossary.

1.6. Visualising STEM Concepts: The Sign Development Process

https://edin.ac/3Wsfhgr

This section will explain the SSC's sign development process. The sign development team consists of individuals from three types of expertise and backgrounds: deaf scientists who hold degrees or PhDs in their respective fields and possess in-depth knowledge of science and STEM; deaf educators/teachers who are proficient in teaching and explaining scientific concepts; and sign linguists, who contribute theoretical insights into signed languages and their linguistic principles. All team members have grown up using sign language, bringing a combination of experiences to the discussions (Cameron et al., 2017; O'Neill et al., 2019). Eight or nine people are invited to join the sign development team for each project. The process begins by examining scientific terms used in the school curricula, which are then grouped into themes or sub-topics. A term is selected, and the team reviews existing signs that may already be in use. If a sign does not exist for a term, it is identified for development.

This sign development process revolves around group discussions, during which team members share their thoughts and ideas on visually representing the term in sign language. Discussions focus on what the term visually represents, its function, and how it should be expressed. Importantly, the process does not centre on the written word or scientific term itself but on creating a sign that visually captures the underlying idea or concept.

For example, consider the term "B-L-A-C-K H-O-L-E". Black holes form when a massive star dies and undergoes a supernova. During a supernova, intense gravitational forces cause the star's core to collapse inward. The written term 'Black Hole' implies something that is 'black' in colour and has a 'hole' in it. However, simply combining the signs "BLACK" and "HOLE" would be inappropriate. Instead, the scientific sign 'BLACK HOLE' visually represents the process of star collapse that leads to the formation of these black holes.

The sign development team carefully considers the meanings of selected terms. Once an idea for a sign is agreed upon, it is captured on video and uploaded to a private MS Team site. Members can then comment, suggest improvements, or approve of the sign. Additionally, deaf children are shown the signs and asked for feedback on clarity.

Once a sign successfully navigates this process, it is refilmed, edited and posted on the SSC BSL glossary website. The website features videos of term definitions, each accompanied by a written English translation, making it a bilingual resource. Visitors can watch the sign and read the text. Pictures that visually match the sign representation have also been added. We also create Example videos to show how the sign is used in sentence. After completion, the content becomes publicly available. A photograph of the BSL glossary team is shown at the end of this video.

BSL Glossary Team

John Denerley
Rob Rattray
Gerry Hughes
Pauline Jordan
Ixone Saenz Paraiso
Mary Frances Dolan
Cathie Birch
Liam McMulkin
Alasdair Grant
Dr Colin Dunlop
Mark McQueen
John Brownlie
Ben Glover
Nicola Jackson
Claire Cummings
Billy Jack Gerrard
Dominic Fox
Derek Rodger
Sanchu Iyer

Tina Kelberman
Dr Audrey Cameron
John Wilson
Gary Quinn
Dr Mark Fox
Lee Robertson
Janet Wardle-Peck
Claire Leiper
Jaabir Mahmoud
Katherine O'Grady
Kirsty Vessey
Ben Fletcher
Ken O'Neill
Malcolm Sinclair
Frankie McLean
Rebecah Taylor
Sujit Sahasrabudhe
Tania Allan

1.7. Subject-Specific Sign Development: Examples from Various STEM Fields

https://edin.ac/4cltNXl

Having explored the general process of sign development, we now turn to specific examples from different STEM disciplines. These examples illustrate how the principles of visual representation and conceptual understanding are applied across various scientific domains. We will examine sign development in Geography, Biology, Astronomy, and Chemistry, each presenting unique challenges and opportunities. Through these examples, we aim to demonstrate how signs are crafted to represent physical landscapes, biological structures, celestial bodies, and abstract chemical concepts. By exploring these diverse fields, we can better understand the versatility and power of sign language in conveying complex scientific ideas.

1.7.1 Geography: Representing Landscapes and Topography

https://edin.ac/4cOAwPw

Geography can be incredibly visual as a subject because the landscape is observable. Signs can show the topography of a place, such as a U-shaped valley or an arête or corrie – caused by glacial erosion. They can also represent rivers. There are many ways that the signs can help us visualise the landscape, including more gently sloping valleys, U-shaped or V-shaped valleys with tributaries feeding into the river. The contours can show how steep the slopes are. On the map, the slopes are steeper if the contours are close together. If they are wide apart, the slope becomes less steep. The shape and movement of the signs can be changed to represent these features and demonstrate the topology. Here are examples. i. topography, ii. map reading (table 2).

1.7.2 Biology: Visual Representation of Location and Function

https://edin.ac/3LvgFZy

Whereas chemistry, as a subject, often deals with abstract concepts (De Jong & Taber, 2007), biology is more visual, involving tangible elements that we can see (Höst, 2022). That said, in developing/creating signs, we still need to think very carefully about where, for example, the organs are in the body, what their actual shape is like, and what their function is.

For example, using the sign for HEART it is necessary to show its location in the body, what it looked like as an organ, where blood flows in (e.g. vena cava) and out (e.g. aorta), and how it fits into the circulatory system. Proper referencing of the sign in the right place is essential to avoid inaccuracies.

The same careful consideration applies to plants. The team spent time examining plant structures and photographs to ensure accuracy. In this picture, you can see the reproductive part of the flower with the stigma and ovules.

Terms like 'stem cell' required particular attention. The signs are a representation of our understanding of a scientific concept. Our bodies contain stem cells (STEM CELL) - these cells have no function other than the capacity. Our initial sign for stem cells was 'STEM CELL', showing the transformation. However, upon consulting with a stem cell research expert, we learned that the initial sign was inaccurate because it depicted differentiation, not the potential of stem cells. Consequently, we revised the sign to accurately represent the potential for change (new STEM CELL sign). We must be mindful of ensuring the signs accurately reflect the location in the body, appearance, and function of the biological organs they represent. Unlike spoken and written language, which can be vague, signs must be precise, as errors are immediately noticeable. The process of creating signs involves extensive discussion, the use of pictures and asking numerous questions to ensure correctness.

1.7.3 Astronomy: Designing Planetary

https://edin.ac/3Ya6Xn4

When thinking about creating signs, we focused on the visual aspects. However, we were also influenced by the character or properties of the objects. An example of this can be seen in the Astronomy in BSL Glossary (Cameron, 2015).

There are eight planets in the Solar System. The first four planets are called the inner planets, followed by an asteroid belt, and the other four are called the outer planets. During the sign development workshops, we created a chart to identify the differences between the planets.

Figure 2: Chart of the solar system planets and their properties. We used the properties in red to help develop the new signs. The planet nearest the Sun is Mercury. The sign we created is MERCURY, which incorporates the sign for HALF. This is because one side of Mercury is incredibly hot, whereas the other is bitterly cold. The side that faces the Sun is exposed to intense heat, whereas the side facing away from the Sun is intensely cold. Mercury rotates very slowly, and due to its little or no atmosphere, it loses any heat that it gains very quickly (NASA Science n.d.).

The next planet out from the Sun is Venus (signs VENUS). The use of a non-manual feature (NMF) (puffed cheeks) represents the incredibly dense atmosphere, which is made up of heavy carbon dioxide gas (CO_2) (NASA Science n.d.).

Next comes Earth, for which there is already a common sign (EARTH) in the deaf community. However, from an astronomy perspective – looking at the Earth from space, we focused on the presence of water (NASA Science n.d.). Therefore, we incorporated the sign for WATER into the sign for EARTH.

Then comes Mars, known as 'The Red Planet ' (NASA Science n.d.). Instead of focusing on its colour, the glossary team chose a different characteristic: Mars has two moons, (MARS). This feature of having two moons is incorporated into its sign.

The next four planets are gaseous giants, while the first four planets are solid and made of rock. This is represented by the closed handshape/fist, which forms the shared base handshape in MERCURY, VENUS, EARTH and MARS. The base sign changes to an open handshape for the four gas giants.

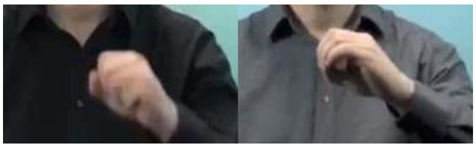


Figure 3: Closed (i) and open (ii) handshapes representing (i) rocky and (ii) gaseous planets.

Jupiter is next (JUPITER), and the open fingers of the right hand represent Jupiter's characteristic stripes (NASA Science n.d.). Saturn (SATURN) follows, with the right hand representing the wide ring system of ice particles with rocky debris and dust circulating the planet (NASA Science n.d.).

Moving along, we come to Uranus. We believe that an Earth-sized object collided with Uranus long ago, knocking it off its original angle of rotation (NASA Science, n.d.). It's the only planet in our solar system that rotates at a 90-degree angle compared to the others. This unique angle of rotation is shown in the sign by changing the orientation of the sign (URANUS). The right hand represents the rings of Uranus, which are smaller than those of Saturn (SATURN).

Finally, we come to Neptune, a very cold planet. Neptune has sixteen known moons, but one (Triton) of them orbits Neptune in the opposite direction to the others (NASA Science, n.d.). This is shown by the 15 moons represented by open fingers moving in one direction and a single finger moving in the opposite direction to represent the moon with the opposite orbit.

These examples demonstrate the sign development team's thinking about visuality and the properties or characteristics when creating the signs. We have tried to create links between.

1.7.4 Chemistry – Representing Abstract Concepts

https://edin.ac/3Sbxnkp

Chemistry is often challenging to teach because it is abstract, conceptual and theoretical, unlike biology, where one can see animals and plans, or physics, where forces and their impacts are observable, such as direction. Chemistry often involves invisible processes (Johnstone, 1991; Taber, 2013; Reid, 2021; Soeharto & Csapó, 2021). For instance, mixing two colourless solutions can suddenly produce a yellow precipitate, but we can't see how or why that happens. Chemistry relies heavily on theoretical concepts and modelling (Taber, 2012).

Extensive research in Chemistry education highlights the importance of visual representation, including drawings, pictures, gestures, and signs, to help students understand these abstract concepts (Gates, 2017; Kiernan et al 2020, 2024). One useful framework is Johnstone's Triangle, which examines chemistry on three levels (Johnstone, 1991; Taber, 2013; Reid, 2021):

1. Macroscopic Level – Observable phenomena. For example, mixing two colourless solutions to form a yellow precipitate or the colour and smell of toast when it pops out of a toaster.

2. Microscopic Level – Atomic and molecular level. This involves visualising ATOMS, PARTICLES and MOLECULES and understanding their interactions, bonding and behaviour.For example, understanding how heating affects bonds within structures.

3. Symbolic Level – Use of symbols and formulas. This includes chemical equations and symbols like 'C' for Carbon; in an equation, '+ O_2' becomes CO_2, representing chemical reactions and compositions.

Teaching Chemistry requires addressing all these levels. Demonstrating experiments and active learning helps link the observable (Macroscopic) with the unseen (Microscopic) and the symbolic representation in equations and formulae. Signs can help to make things clearer at the microscopic level (Clark et al., 2021; Cameron et al., 2017).

For example, the sign for ATOM involves moving the index finger of the dominant right hand around a closed left fist, representing an ELECTRON orbiting a nucleus. Using a single finger in the horizontal plane looks like the symbol for 'negative' (-) and indicates that the electron has a negative charge. The closed left fist in the sign represents the nucleus with a positive charge, which attracts the negatively charged electron, preventing it from leaving its orbit. We don't know where the electrons are at any given moment, but you can find them within the orbitals. This sign can be modified to show different orbital structures, such as s- and p-orbitals. In general, we use the ATOM sign to show the movement of electrons, and as one studies further, the shape of different orbitals can be modified.

The NUCLEUS has two types of particles: NEUTRONS, represented by a symbol for zero due to their neutral charge, and PROTONS, which is positively charged. Within the nucleus, we have NEUTRONS and PROTONS. Combined with the ELECTRONS, we have the sign for ATOM.

The sign ATOM is a visual representation that can be further modified to show, for example, the exchange of electrons between atoms in a CHEMICAL REACTION such as a REDOX REACTION. This sign illustrates the loss of an electron from one atom (OXIDATION) and the gain of an electron by another atom (REDUCTION) during the reaction. Research indicates that such visual models are crucial for understanding

Chemistry, and the flexibility these signs offer can aid that understanding. Representing theoretical ideas through pictures, models, experiments, and written formulae aids in comprehending this complex subject.

1.7.5 Family of Signs: Aiding Conceptual Understanding

https://edin.ac/4bTtck6

The sign development team thought long and hard about creating links between the signs we created to form what we call a 'family' of signs, which together build an understanding of the broader concept (Cameron et al., 2017; Quinn et al., 2021). For example, consider the terms 'Mass' and 'Weight'. We already have a sign in common use, WEIGHT, but the sign is different in the context of STEM. We focused not on the deaf community's everyday use of the word 'Weight' but on its scientific context. 'Mass' differs from 'Weight'; everything is made up of mass - my clothing, body, the air around us, tables, etc. 'Weight', on the other hand, is a force acting on mass, and that force is 'Gravity'. The sign for mass is this (*signs MASS*), which represents the matter from which everything is made, and this is the sign for GRAVITY. If we combine these two signs (MASS + GRAVITY), we get WEIGHT, which demonstrates the concept: MASS + GRAVITY = WEIGHT.

By using the sign, we can show the force of gravity, for example, here on Earth, compared to on the moon, where the gravitational pull is less. In this instance, the sign is articulated more slowly – we see the impact on weight in the film of astronauts bouncing along on the surface of the moon. What's important to note is that the mass remains the same here and on the Moon – the gravitational pull is different: GRAVITY (on Earth) and GRAVITY (on the Moon). Children understand this when they see the signs, and that's significant.

Another example of a 'family' of signs is in Chemistry, specifically within the theme of chemical reactions. Take a look at the following different chemistry terms and variations in signing.

CHEMICAL REACTION - REACTANT - PRODUCT - NON-REVERSIBLE REACTION - REVERSIBLE REACTION - ENDOTHERMIC REACTION - EXOTHERMIC REACTION

These are all different terms, yet they are connected to one another. The morphology of the sign varies slightly to represent each chemistry term, but notice that every sign has the same central movement (from left to right), and each introduces a slight variation to create a new meaning/product as the sign concludes on the right. The movement follows the direction of a written chemical equation – this left-to-right movement is maintained throughout. Movement can also be modified to represent the speed of the reaction, whether it be fast or slow. It can also be modified to show the presence of heat being given off – EXOTHERMIC, or a drop in temperature occurring during the reaction – ENDOTHERMIC. Throughout all of these modifications in the morphology, the placement and movement are maintained – creating a connection between signs, which helps to construct a sound conceptual understanding.

1.8. Impact of Sign Glossaries on Learning

1.8.1 Conceptual Understanding: Electricity – AC vs DC Concept

https://edin.ac/46eUm3J

Does access to the STEM Sign Glossary aid conceptual understanding? In a small research project using a linguistic ethnographic methodology to explore the use of sign language in dialogue, I had the opportunity to observe and video record pupil discussions in the science classroom (Kusters & Hou, 2020). One part of the process that drew particular attention was a group of pupils who had been instructed to research and prepare a presentation on the differences between AC (alternating current) and DC (direct current) as part of the topic on 'electricity'. The students went home and returned the following day with their presentations.

While presenting their research to their peers, the first student referred to the word 'current' on their slides. At this point, the student seemed a little uncertain about the meaning of the word and signed CURRENT in its more usual sense, meaning 'now' or 'currently' – clearly a different meaning. The glossary includes signs for 'Current', 'AC' and 'DC' to represent the flow of electricity along a wire; DC represents the electrical current flowing in one direction only, while AC represents current flowing in both directions. This pupil didn't know these signs, leading to an incomplete understanding of the concept in this context and signing CURRENT as 'now/currently'.

The pupil then asked the class teacher for the sign for the scientific term, and when the teacher signed CURRENT, the pupil understood. The next student to present used the sign (electrical) CURRENT correctly. This demonstrates the importance of having the appropriate sign as an aid to understanding (Lang et al., 2007; Kurz & Pagliaro, 2019; Enderle et al. 2020; Cameron, 2024).

1.8.2 Vocabulary Access: Teaching Density

https://edin.ac/3zOQISb

A different group activity illustrates how using a sign glossary can improve conceptual understanding (Cameron, 2024). In a class of younger pupils (five to six years old) learning about the terms 'float' (FLOAT) and 'sink' (SINK), the teacher instructed the children to collect objects and drop them into a tank of water, predicting whether each item would float or sink.

One pupil picked up an object, dropped it into the tank, and was surprised when it floated instead of sinking. Throughout the activity, some items floated while others sank. After the activity, the teacher explained that whether an object floats or sinks is determined by 'density' (DENSITY), a sign from the SSC glossary. The teacher clarified that if an object is dense, it will sink; if it is not, it will float. Density is a combination of mass and volume - objects with less mass relative to their volume will float, while those with more mass (particles closely packed together) will sink, like metal. However, with more space between particles, wood will float, as seen when a tree trunk falls in water.

The teacher explained these concepts to the five- and six-year-olds. At the end of this lesson, the teacher asked again, "Why do some objects float and others sink?". They responded using the sign DENSITY from the glossary, demonstrating that access to the sign glossary and vocabulary in sign language can aid conceptual understanding.

1.8.3 Facilitating Understanding Through Sign and Dialogue

https://edin.ac/3zVzYse

Lindahl (2015 & 2021) found that sign language, along with text and pictures, can facilitate access to conceptual understanding. These elements help, and signs are particularly important as part of the dialogue. Lindahl discovered that while access to sign vocabulary was important, it was not sufficient on its own. More is needed to facilitate discussion and the construction of meaning. Lindahl also emphasises the importance of teachers understanding these signed discussions so that they can recognise when pupils are using signs that indicate their understanding.

1.9. Conclusion

https://edin.ac/3Y8PkDX

The development and implementation of sign language glossaries for STEM subjects represent a significant advancement in deaf education. As we have seen throughout this chapter, the SSC BSL Glossary project and similar initiatives worldwide are not merely about translation; they are about creating visual representations that capture the essence of scientific concepts. The process of developing these signs involves deep consideration of scientific principles, visual representation, and linguistic structures, resulting in a powerful tool for conceptual understanding. The examples from geography, biology, astronomy, and chemistry demonstrate how well-designed signs can make abstract concepts more tangible and accessible. Moreover, the observed impacts on classroom learning underscore the importance of these resources. As we move forward, continued research, collaboration between deaf scientists, educators, and linguists, and the integration of sign language resources into STEM curricula will be crucial in ensuring equal access to scientific knowledge for deaf students. This work not only enhances education for deaf learners but also enriches the field of science communication as a whole, demonstrating the unique power of visual language in conveying complex ideas.

Table 1: Existing Global STEM Sign Language Dictionaries/Glossaries/Lexicons

**These lexicons have STEM within a large lexicon (not solely for STEM).*

Name	Country	Language	Website
AfricaSign	Africa	Various	https://www.africa-sign.org/
ASL-Clear	USA (Framingham)	American Sign Language	https://aslclear.org/
ASL-Core	USA (Rochester)	American Sign Language	https://aslcore.org/
ASL-STEM	USA (Washington)	American Sign Language	https://aslstem.cs.washington.edu/
Astronomy	France (book)	French Sign Language	http://sion.frm.utn.edu.ar/iau-inclusion/wp-content/uploads/2017/11/Dictionnaire-Frances.pdf
Atomic Hands	USA	American Sign Language	https://atomichands.com/
Austin Community College	USA (Austin)	American Sign Language	https://accmultimedia.austincc.edu/signs/
British Sign Language Glossaries of Curriculum Terms	United Kingdom (Edinburgh)	British Sign Language	https://www.ssc.education.ed.ac.uk/BSL/
Chemistry for High School Students; Computer Science	Greece	Greek Sign Language	https://prosvasimo.iep.edu.gr/el/gia-mathites-me-provlimata-akohs/xhmeia-me-nohma-b-g-gymnasiou-gia-kofous-kai-varikoous-mathites?fbclid=IwAR0k1UAwuCXs4M67v7vwv6cVf-WtHJrDk7iRlYdVccDK1HRlL_SzqqZxi2Q (available on CD-ROM)
Cité des Sciences (Museum)	France (Paris)	French Sign Language	https://www.cite-sciences.fr/fr/ma-cite-accessible/sourds-et-malentendants/ressources/signaire-lsf/
DeafTEC	USA (Rochester)	American Sign	https://deaftec.org/stem-dictionary/about-the-project/

Dictio*	Czech Republic	Many sign languages	https://www.dictio.info/
Elix*	France	French Sign Language	https://dico.elix-lsf.fr/
Greek Sign Language*	Greece	Greek Sign Language	https://www.ocean.upatras.gr/gsl/
GEIL Libras Study and Innovation Group (Pontifical Catholic University of Rio Grande do Sul)*	Brazil (Porto Alegre)	LIBRAS: Brazilian Sign Language	https://www.youtube.com/channel/UCZZtQOxbvuWdNhbJ_a5bq2g/playlists
ISLEVL - Indian Sign Language Enabled Virtual Lab	India (Chandigarh)	Indian Sign Language	https://islevl.org/
INJS Bourg-la-Reine	France	French Sign Language	INJS Bourg-la-Reine https://ijs.92.dico.free.fr/maths/index.html
INSA (civil engineering)	France (Toulouse)	French Sign Language	http://devv4.insa-toulouse.fr/fr/formation/glossaire-gc-en-lsf.html
Irish Sign Language STEM Glossary	Ireland (Dublin)	Irish Sign Language	https://www.dcu.ie/islstem
Les Doigts Qui Rêvent (geology)	France (Dijon)	French Sign Language	ttps://ldqr.org/mots-de-geologie-en-lsf/
LexiQue	Canada (Quebec)	LSQ	https://lexiquelsq.ca/theme/science-et-technologie/
Madrasati Signs Platform	Morocco	Moroccan Sign Language	https://madrasati-signs.org/
New Zealand Sign Language Dictionary*	New Zealand	New Zealand Sign Language	https://www.nzsl.nz/
Ocelles*	France (Paris)	French Sign Language	https://ocelles.inshea.fr/fr/accueil

Projeto Surdos - UFRJ	Brazil (Rio de Janeiro)	Libras	https://www.youtube.com/@projetosurdos/playlists
Quantum ASL	USA (Harvard University)	American Sign Language	https://www.youtube.com/channel/UC3etnnslxGpH89XgojqE0Ng
Shuwaemon	Japan	Japanese Sign Language	www.shuwaemon.org
Sign2MINT	Germany	German Sign Language	https://sign2mint.de/
Sign "Maths"	France (Toulouse)	LSF	signmaths.univ-tlse3.fr
SignBank*	Australia	Auslan	https://auslan.org.au/
Signing Science and Math Dictionaries	USA (Cambridge)	American Sign Language (Avatar)	https://signsci.terc.edu/index.html
Slovnik	Czech Republic (Brno)	Czech Sign Language	https://slovnikczj.vutbr.cz/
Spread the Sign*	Global	Different sign languages	https://www.spreadthesign.com/en.gb/search/
STIM Sourd France	France	LSF	www.stimsourdfrance.org
Texas Math Sign Language Dictionary	USA (Texas)	American Sign Language	https://www.texasdeafed.org/Page/516
UVED (sustainable development)	France (Toulouse)	French Sign Language	https://www.uved.fr/fiche/ressource/glossaire-du-developpement-durable-en-langue-des-signes-francaise-lsf

Complied by the Global STEM sign language lexicon team in 2023.

Table 2: STEM Signs in BSL

Sign	Source
Alternating current	https://www.ssc.education.ed.ac.uk/BSL/physics/alternating.html
Aorta	https://www.ssc.education.ed.ac.uk/BSL/environment/aorta.html
Arête	https://www.ssc.education.ed.ac.uk/BSL/geography/arete.html#start
Asteroid belt	https://www.ssc.education.ed.ac.uk/BSL/astronomy/asteroidbelt.html
Atom	https://www.ssc.education.ed.ac.uk/BSL/chemistry/atom.html#start
Atria/atrium	https://www.ssc.education.ed.ac.uk/BSL/environment/atria.html
Black hole	https://www.ssc.education.ed.ac.uk/BSL/astronomy/blackhole.html
Bonding	https://www.ssc.education.ed.ac.uk/BSL/chemistry/bond.html#start
Carnivores	https://www.ssc.education.ed.ac.uk/BSL/environment/carnivores.html
Chemical reaction	https://www.ssc.education.ed.ac.uk/BSL/chemistry/chemreact.html#start
Circulatory system	https://www.ssc.education.ed.ac.uk/BSL/environment/dualcirculatorysystem.html
Contours	https://www.ssc.education.ed.ac.uk/BSL/geography/contours.html#start
Corrie	https://www.ssc.education.ed.ac.uk/BSL/geography/corrie.html#start
Current	https://www.ssc.education.ed.ac.uk/BSL/physics/current.html#start
Density	https://www.ssc.education.ed.ac.uk/BSL/physics/density.html
Differentiation (stem cell)	https://www.ssc.education.ed.ac.uk/BSL/biology/differentiation.html#start
Direct current	https://www.ssc.education.ed.ac.uk/BSL/physics/directcurrent.html#start
Distillation	https://www.ssc.education.ed.ac.uk/BSL/chemistry/distillation.html#start

Earth	https://www.ssc.education.ed.ac.uk/BSL/astronomy/earth.html
Electricity	https://www.ssc.education.ed.ac.uk/BSL/physics/electricity.html
Electron	https://www.ssc.education.ed.ac.uk/BSL/chemistry/electron.html#start
Endothermic reaction	http://www.ssc.education.ed.ac.uk/BSL/chemistry/endothermic.html#start
Exothermic reaction	http://www.ssc.education.ed.ac.uk/BSL/chemistry/exothermic.html#start
Giant planet	https://www.ssc.education.ed.ac.uk/BSL/astronomy/giantplanet.html
Glacier	https://www.ssc.education.ed.ac.uk/BSL/geography/glacier.html#start
Gravitational pull	http://www.ssc.education.ed.ac.uk/BSL/physics/gravitational.html
Gravity	http://www.ssc.education.ed.ac.uk/BSL/physics/gravity.html#start
Inner planets	https://www.ssc.education.ed.ac.uk/BSL/astronomy/innerplanets.html
Interdependence	https://www.ssc.education.ed.ac.uk/BSL/environment/interdependence.html
Interdependence	https://www.ssc.education.ed.ac.uk/BSL/environment/interdependence.html
Interdependence definition	https://www.ssc.education.ed.ac.uk/BSL/environment/interdependenced.html
Jupiter	https://www.ssc.education.ed.ac.uk/BSL/astronomy/jupiter.html#start
Map	https://www.ssc.education.ed.ac.uk/BSL/geography/map.html#start
Mars	https://www.ssc.education.ed.ac.uk/BSL/astronomy/mars.html#start
Mass	http://www.ssc.education.ed.ac.uk/BSL/physics/mass.html
Mass	https://www.ssc.education.ed.ac.uk/BSL/physics/mass.html#start
Mercury	https://www.ssc.education.ed.ac.uk/BSL/astronomy/mercury.html#start

Mixture	https://www.ssc.education.ed.ac.uk/BSL/chemistry/mixture.html#start
Molecule	https://www.ssc.education.ed.ac.uk/BSL/chemistry/molecule.html#start
Moon	https://www.ssc.education.ed.ac.uk/BSL/astronomy/moon.html
Neptune	https://www.ssc.education.ed.ac.uk/BSL/astronomy/neptune.html#start
Neutron	https://www.ssc.education.ed.ac.uk/BSL/chemistry/neutron.html#start
Non-reversible reaction	http://www.ssc.education.ed.ac.uk/BSL/chemistry/nonrevers.html#start
Nucleus	https://www.ssc.education.ed.ac.uk/BSL/chemistry/nucleus.html#start
Outer planets	https://www.ssc.education.ed.ac.uk/BSL/astronomy/outerplanets.html#start
Ovules	https://www.ssc.education.ed.ac.uk/BSL/biology/ovules.html
Particle	http://www.ssc.education.ed.ac.uk/BSL/chemistry/particle.html#start
Planet	https://www.ssc.education.ed.ac.uk/BSL/astronomy/planet.html
Product	http://www.ssc.education.ed.ac.uk/BSL/chemistry/product.html#start
Properties	https://www.ssc.education.ed.ac.uk/BSL/chemistry/properties.html#start
Proton	https://www.ssc.education.ed.ac.uk/BSL/chemistry/proton.html#start
Reactant	http://www.ssc.education.ed.ac.uk/BSL/chemistry/reactant.html#start
Reflecting telescope	https://www.ssc.education.ed.ac.uk/BSL/physics/reflecting.html
Reversible reaction	http://www.ssc.education.ed.ac.uk/BSL/chemistry/reversible.html#start
River	https://www.ssc.education.ed.ac.uk/BSL/geography/river.html#start
Saturn	https://www.ssc.education.ed.ac.uk/BSL/astronomy/saturn.html#start
Sign	Source

Solar system	https://www.ssc.education.ed.ac.uk/BSL/astronomy/solarsystem.html#start
Stamen	https://www.ssc.education.ed.ac.uk/BSL/biology/stamen.html#start
Stem cell	https://www.ssc.education.ed.ac.uk/BSL/biology/stemcell.html#start
Stigma	https://www.ssc.education.ed.ac.uk/BSL/biology/stigma.html
Topography	https://www.ssc.education.ed.ac.uk/BSL/geography/topography.html#start
Tributary	https://www.ssc.education.ed.ac.uk/BSL/geography/tributary.html#start
Uranus	https://www.ssc.education.ed.ac.uk/BSL/astronomy/uranus.html
U-shaped valley	https://www.ssc.education.ed.ac.uk/BSL/geography/ushapedvalley.html#start
Vena cava	https://www.ssc.education.ed.ac.uk/BSL/environment/venacava.html
Ventricles	https://www.ssc.education.ed.ac.uk/BSL/environment/ventricles.html
Venus	https://www.ssc.education.ed.ac.uk/BSL/astronomy/venus.html
V-shaped valley	https://www.ssc.education.ed.ac.uk/BSL/geography/vshapedvalley.html
Weight	http://www.ssc.education.ed.ac.uk/BSL/physics/weight.html#start

2.The Development of Mathematical Skills of Deaf Learners: Insights from Research and Examples from Practice:

Olga Pollex, Swetlana Nordheimer, Viktor Werner

Research on the topics of mathematical development and teaching of deaf children has a long and complex tradition with very diverse theoretical, empirical and school practical approaches to education of deaf learners (Fleri, 1835; Tabak, 2014, 2016, Marschark & Knoors, 2012; Werner et al., 2019; Hänel-Faulhaber et al., 2023). To give an idea of this variety this chapter refers to the theoretical approaches established and grounded in empirical studies. To modify existing theoretical approaches by considering new findings in the research and expectations of professionally prepared teachers of mathematics we seek to refer to more recent empirical studies focused on mathematical education of deaf schoolchildren. These studies give evidence for positive effects on teaching mathematics in Sign Languages. Closing our considerations by concrete examples from deaf mathematical classroom we would like to challenge experts from research and practice with new ideas and open questions. We aim to be critical in theoretical research and concrete in our suggestions for school practice, which, according to Becker (2019, p. 85), is sometimes ahead of educational policies.

We, the authors of the paper, come from different theoretical traditions and use different research methods in our scientific work. This article should therefore be understood as a multi-perspective dialog. Educational researchers, teachers, parents and other stakeholders are invited to take part in it.

Readers who have a basic knowledge of Deaf Studies and pedagogy of Sign Languages but are not familiar with didactics of mathematics will be hopefully pleased to gain some insights into trends in mathematics education. Those who are already familiar with works at the intersection of Sign Languages pedagogy and the didactics of mathematics can take the opportunity to expand their knowledge and evaluate presented research findings and innovative teaching methods. The handbook is an invitation to explore a complex set of phenomena for which there is no single theoretical explanation. However, there are some exciting theoretical, empirical and practical approaches to investigation into learning of mathematics in Sign Languages. Our paper aims to show how seemingly different research perspectives can complement each other and that progress towards a comprehensive account of deaf children's mathematical abilities and appropriate fostering of potentials requires a broad understanding of research from more than one perspective and discipline. The main focus of the paper is to provide arguments for mathematics education in Sign Languages and to give some ideas how it could be done.

In line with the diversity of positions, theoretical backgrounds and practical intentions in mathematical education of deaf schoolchildren, the terms “deaf” or “hard of hearing” are not used consistently in the scientific literature (see Szűcs, 2019, p. 3). This makes it difficult to interpret and reflect on the results. To avoid misunderstandings, we will use the term “deaf” as in Scott et al. (2023, p. 3) “to refer to a range of hearing levels, from what might typically be referred to as hard-of-hearing, to profoundly deaf; we also include anyone who would benefit from being identified as deaf such as those with central auditory processing disorder, as we believe that all would benefit from the model proposed here.” However, the focus of this paper is on deaf learners whose main mode of communication is one or more Sign Languages.

We will start with theoretical framework for learning of mathematics in sign languages and move on to the empirical studies focused on the effects of Sign Languages on learning of deaf schoolchildren. We then present selected intervention studies (Nunes & Moreno 1998, 2002, Nunes 2004, Wille 2018, 2019, 2020, Angeloni & Wille 2022) and practical examples as a source for arguments for signed mathematics on the one hand, and as a source for inspiration for development of didactical concepts and materials on the other hand. Finally, we will introduce the following chapters of the handbook, which deal specifically with the theory and practice of teaching algebra, stochastics and geometry with deaf children. We make no claim to completeness of our work and look forward to constructive criticism of our readers, but we are sure that we address important aspects of learning of and learning in Sign Languages in mathematics lessons. Our investigations can be deepened by the readers through recommended scientific literature or direct contact with the teachers and researchers we mentioned here and last but not least with us.

Theoretical Framework

Looking at the contributions to the conference of the “Gesellschaft für Didaktik der Mathematik” in Englisch “Society for Didactics of Mathematics“ (GDM) in Germany since 2010, we can see that interest in Sign Languages is growing in the German-speaking mathematics didactics community. While no articles on teaching mathematics in sign languages were published in the years between 2010 and 2015, seven were published in the years 2016 to 2021 and seven in the last three years 2022 to 2024. On the other hand, the number of articles relating to the subject of mathematics in the journal *DAS ZEICHEN*, German Journal for Language and Culture of the Deaf, has also increased, particularly in recent years. In 2022, for example, there were three articles on the topic centered on the sign languages in mathematic lessons.

The positive effects of Sign Languages on the learning of mathematics and mathematical development of deaf learners have been acknowledged in the history of Deaf education and intensively studied for example by Rosanova (1978) and Yashkova (1988). They dealt with the influence of languages on the organization of knowledge in memory and mathematical problem-solving process in deaf learners.

Sign languages and organization of knowledge in memory and problem solving

In the studies of Rosanova (1971, 1978) and Yashkova (1988) it was empirically proven that deaf children are multilingual and that different language systems are linked in their thinking in a complex way. Rosanova (1971) showed that deaf learners grouped gestures and signs into semantic fields much more easily than words and were thus better able to retain them. The number of grouped gestures, signs and words increased over the course of schooling. At the same time, deaf learners became better at grouping not only gestures and signs but also words as they got older. With the age of the learners, the precision with which the learners assigned the signs and words to each other according to their meaning also increased. This means that the memory content of deaf people is organized differently from that of hearing people due to their multilingualism. Recent studies conducted by Villwock et. al. (2021) give differentiated, deep and empirically grounded insights into complexity of activation of different languages by hearing and deaf ASL-English bilinguals when they process written words. These results are not obtained with specifical mathematical terms, but they are still relevant for mathematical teaching of deaf learners especially when it comes to the so-called text-problems or word-problems. They also give evidence for the actuality and relevance of the didactical assumptions of empirically described multilingualism made by Rosanova (1971).

Model of thinking development inspired by Rosanova (1978) and Yashkova (1988)

The studies by Rosanova (1978, 1991) showed that development of language competences is a very important factor in the mathematical development. However, Rosanova's data showed empirically that language competences alone are not decisive for the successful development of mathematical abilities in deaf school children. To understand the theory behind empirical studies we will now refer to theoretical approaches developed by Rosanova (1978) and Yashkova (1988) and explain the difference between so called **visual-imaginative** and **logical-verbal** thinking as these terms are used by Rosanova (1978).

Visual-imaginative thinking is the ability to think in images and representations that replace real objects in order to carry out mental operations. Here not only the external appearance, but also the properties of objects and relationships between them should be taken into account. To this end, Rosanova (1978) recommends strengthening the relationships between objects and words that denote these objects, their properties and relationships. We go further and suggest that the development of visual-imaginative thinking can be mediated, guided, supported and strengthened by the use of productive and conventionalized signs and gestures as designations of mathematical objects, mathematical objects themselves, their properties and relationships between them, which can be confirmed by empirical data obtained by Rosanova (1971, 1978) and Yashkova (1988). Following Rosanova (1978), the term “visual-imaginative” is used here to emphasize that it is not only about the sensorial perception and operation with visible objects and images, but also about the mental imagination, the imagining of objects, their properties, structures and operations with them and the detachment from the visibly perceptible objects and models to operate with mental and not necessary perceivable (for example visible or tangible) with physical eyes or hands images.

Logical-verbal thinking involves formal mental operations mediated through language that may be completely detached from real objects. Here, too, we go further than Rosanova (1978) and suggest that this form of thinking should also be consciously embedded in Sign Languages as early as possible in order to provide optimal teaching and support. At the same time with conventionalized symbolical signs presented objects and mental operations can be accompanied by written language and, if desired and required, by spoken languages. The decisive factor here is that the individual sensorial abilities to perceive symbols, individual linguistic repertoires and current stage of mathematical development with regard to visual-imaginative and verbal-logical thinking of learners are specifically taken into account.

Culturally established national Sign Languages are not the only forms of communication observed between parents and their deaf children. Morford (1996) investigated **Home Sign** as a variant of signing which is often used in families where hearing parents have deaf children. Normally the signs in this language do not extend beyond the family and are initially mostly pictorial or natural signs. When all family members learn these signs, the pictorial component of sign communication is reduced, as the pictorial nature of the signs according to Morford (1996) does not facilitate learning and memorization. In families where the parents do not speak the official Sign Languages of their country, the Home Sign as symbolic language plays a very important role. According to Morford (1996), when the critical period for language acquisition has arrived, the child begins to perceive the parents' natural signs as language information, memorizes them and uses them later to communicate with them. However, the vocabulary at home is usually limited and cannot be used in communication outside the family.

Interdependence between visual-imaginative and verbal-logical thinking

Visual-imaginative thinking as thinking about visually perceivable objects and models or operating with mental images and logical-verbal thinking as operating with linguistic and symbolic tools are mutually **interdependent** in development. Visual representations and images are not self-explanatory and can only be interpreted in the context of linguistical or other symbolic explanations or experiences. This means that preschool children are not only able to interpret visually perceptible pictures in children's books but can also understand linguistic-logical relationships in stories, even if they are not presented with visible pictures, but told or signed to them about. According to Loots et al. (2005), hearing parents who use Sign Language to communicate with their young deaf children are much more successful in involving their children in communication with the help of symbolic language than parents who prefer purely verbal communication (see also Rathmann et al. 2007). Parents who practice total communication (all possible ways of conveying information with extensive use of natural signs) come close to those who use signs, but they still lag behind the group of parents who use sign language in terms of success in exchanging symbolic language categories with their children (Khokhlova, 2013).

As early as 1965 Vernon (2005) has pointed out the heterogeneity of the group of deaf children with regard to their cognitive development. He stated that the results of deaf children in comparison to hearing children depend on the test methods, qualifications and expertise of professionals who are responsible for the diagnosis. Rosanova (1978, 1991) and Yashkova (1988) also emphasized that the thinking preferences and abilities of deaf children vary greatly from individual to individual. Rosanova's results showed that even in the non-verbal tests, the deaf children whose visual-imaginative and logical-verbal thinking were harmoniously developed performed better. This must be taken into account in the classroom. In addition to fostering language abilities and skills in the context of mathematical problem solving, Rosanova (1971, 1978, 1991) recommended targeted support for visual-imaginative thinking or the ability to visualize mathematical content.

Yashkova (1988), who had also empirically studied the development of mathematical thinking in deaf children, described a model for the development of mathematical learning in deaf children, in which visual-imaginative and verbal-logical thinking were seen as integral parts of the developmental process. Yashkova's (1988) concept had taken gestures and signing into account, but the focus of her research was on the fostering of spoken language. Since gestures and signs were not excluded from the study, Yashkova's (1988) model can be modified for teaching mathematics in Sign Languages and can become a structuring element on the one hand and the subject of future didactic research on the other.

According to Yashkova's model (1988), even in the early stages of development and when solving mathematical problems by operating with objects, the success of mathematical development depends on the extent to which children's practical activities are embedded in language. Language also plays a special role in the transition from operating with models to visual-imaginative thinking which leans not on models but on their visual or mental representations. It helps learners to detach themselves from concrete objects and to use their pictorial and schematic representations of objects, conveyed by signs and gestures, as a basis for reasoning. At first, the deaf learners can solve difficult problems with the help of practical actions and, if necessary, with the help of adult signers, then, with increasing experience, learners develop rational solutions and can express them in Sign Languages in their own independent way. Later on, mathematical arguments can already be found in the form of visual representations of objects, on the basis of pictures with actions and described with the help of productive signs as well as mathematical conventionalized signs.

An important prerequisite for the development of visual-imaginative thinking is the development of the ability to differentiate between plans of real objects and models that reflect these objects. To this end, the generalization and schematization of pictorial representations can first be practiced through Sign Languages, then the transitions to the next stages of generalization of images and more complex schemata. Sign languages allow the detachment from concrete objects and their pictorial representations by enabling the operation with mental images. The models of Rosanova (1978) and Yashkova (1988) are based on their extensive experimental studies, in which quantitative and qualitative research methods were combined. But what can we learn from more recent empirical studies which focus on the use of Sign Languages in mathematical teaching?

Empirical Findings

To investigate new knowledge about processing of Sign Languages and signed numbers in the brain of Sign Language users, psycholinguistic methods and neuroscience are used in more recent studies.

Neurological findings on language processing

Neville et al. (1998) used functional magnetic resonance imaging (fMRI) to show which areas of the brain were activated during the processing of written English or American Sign Language (ASL) in deaf signers, hearing signers and hearing non-signers. It was found that sign languages were processed differently from written language. All groups, hearing and deaf participants, with English or ASL as their preferred communication modality, showed strong and repetitive activation in the left hemisphere and thus in the brain areas commonly associated with language processing. In addition, hearing and deaf participants who were Sign Language oriented showed extensive activation in the right hemisphere, suggesting that the specific demands of language also partly determine the organization of language systems in the brain. Masataka et. al (2006) focused on the processing of signed numbers by deaf sign language-oriented individuals and proposed that "In all, the network exists on a non-linguistic basis and functions for the retrieval of arithmetic facts from presented linguistic material regardless of the mode of the language, that is, a region of parietal cortex underlies an abstract-semantic number sense, and a region of left prefrontal cortex underlies more specific operations mediating exact or approximate calculation. Particularly, the fact that linguistic representations of exact numerical values are controlled in the brain's left hemisphere even in native signers should be intriguing." However, we still know too little about the functions of the brain to directly derive from these concrete didactic consequences for the planning of teaching processes (Becker, 2006). For this reason, in the next step we will turn to more recent findings from the developmental psychology of deaf children.

Importance of early language support for mathematical development

Khokhlova (2013) and Bogdanova (2021) summarized recent studies on the role of Sign Languages in the communicative, cognitive and social development of deaf and hard of hearing children. They found that a number of studies have shown that deaf children of deaf parents are not inferior to hearing children in terms of their cognitive abilities and that the mastery of Sign Language positively influences the cognitive development of deaf children. Sign Languages promote creativity in deaf children, lead to a better understanding of spatial relationships and to greater flexibility in problem solving.

Many researchers recognize the need for early acquisition of sign language by deaf children. Sign languages can serve as a linguistically symbolic means of communication, which is crucial of the first stages of children's development and contribute to the development of the cognitive and personal domain by creating the conditions for emotional well-being. Based on the studies, Bogdanova (2021) points out the challenges of diagnostics in sign languages. This is exactly where the work of Werner and Hänel-Faulhaber (2023) comes in. They are developing tests which are appropriate for deaf children.

Werner and Hänel-Faulhaber (2023) investigated the understanding of repeating patterns in deaf and hearing children. The children had to fill in a gap in the patterns. It was found that the solution scores of deaf children who learned Sign Languages at an early age are comparable to those of hearing children. In contrast, deaf children who learned Sign Languages later are less successful. This shows that Sign Language has a positive effect on solving pattern tasks. Earlier studies by Werner (2010) and Werner et al. (2019) also indicate that Sign Language support has a positive effect on the mathematical development of younger deaf children.

Santos and Cordes (2022) showed in their studies that deaf children who are not exposed to fluent language from birth generally lag behind their hearing peers in mathematics. These inequalities occur as early as the age of 3 and can persist into adulthood (Kramer & Grote, 2009). The empirical data obtained by Santos and Cordes (2022) suggest that limited access to language, especially in the first months of life, may create a risk to the acquisition of early number concepts and mathematical problem-solving skills. The study focuses on the role of the working memory of deaf children in mathematical learning. These results are consistent with the findings of Walker et al. (2024) who, in a study with 188 children aged 4.5 to 9 years, discovered the relationship between language experiences and children's ability to match number signs or number words to Arabic numeral symbols and cardinal numbers. The results suggest that early access to language, whether spoken or signed, supports the development of age-typical mapping skills and that knowledge of number words is crucial for this development.

An evaluation of a version of the mathematical diagnostic test MBK 0 (test of basic mathematical skills at kindergarten age; Krajewski, 2018) in German Sign Language (DGS) found that the results of six-year-old deaf native signers correspond to the (hearing) age norm (Werner & Hänel-Faulhaber, 2024).

Sensory experiences through the use of hands when counting

The potential of sign language-based mathematical support also arises from the fact that the use of sign numbers and sign algorithms allows new sensory experiences. For example, Di Luca and Pesenti (2011) have shown that the representation of numbers as finger-configurations offers children the opportunity to learn and internalize basic properties of natural numbers through sensorimotor interactions with the world. Recent findings show that adults also use their fingers as a visuomotor support to process, represent and communicate numbers, regardless of their hearing status and educational background. It has been shown that the use of fingers to prototypically represent numbers gives the corresponding finger configurations a special status in long-term memory: these configurations are recognized and processed faster than other finger configurations and provide direct access to number size, which other finger configurations do less efficiently.

Di Luca and Pesenti (2011) argue that finger-numbers help to acquire, build and then access number semantics, and that they provide additional value compared to other number representations by anchoring the meaning of numbers in a culturally shared but non-arbitrary and self-experienced sensory-motor representation.

At this point, however, it is important to note that finger counting as described by Di Luca and Presenti (2011) is fundamentally different from the counting systems that are integrated into different national Sign Languages as a part of mathematical cultural heritage (Fleri 1835, Rainò et. al., 2018). Similar to finger counting, signed numbers allow sensory experiences and support short- and long-term memory. But in contrast to finger counting, they represent complex mathematical symbols and algorithms (Rainòet. al. 2018, Werner & Hänel-Faulhaber, 2024).

Importance of counting and calculation algorithms in sign languages

Leybaert and van Cutsem (2002) investigated to what extent the visual-manual modality and the structure of the sign number sequence has an influence on the development of counting and its use by deaf children. For example, the number sequence in Belgian French Sign Language follows a base-5 rule, while the number sequence in oral French follows a base-10 rule. To illustrate this special characteristic of Sign Languages, we would like to draw your attention to the project "Nina im Zahlenland" (Nina in Numberland): https://ksl-msi-nrw.de/de/node/5134 , which was created by the team at TU Dortmund University (Math inclusive with PIKAS) and the University of Hamburg (MaBaKo-Deaf) with scientific support from Viktor Werner. The signed numbers from 1 to 100 can be found there.

The numbers 11 to 20 in particular differ in various dialects of German Sign Language. While the NRW (Nordrhein-Westfalen) variants are used in the "Nina im Zahlenland" project, numbers are signed differently in Berlin. The representation of numbers in German Sign Language and dialectal differences are discussed in more detail in Papaspyrou et. al (2008). Numbers are also signed differently in different national Sign Languages, whereby base 5 is retained. For example, Ukrainian Sign Language (UGS) also works with base 5. However, base 5 is represented differently in UGS than in DGS (German Sign Language), (see Figure 1).

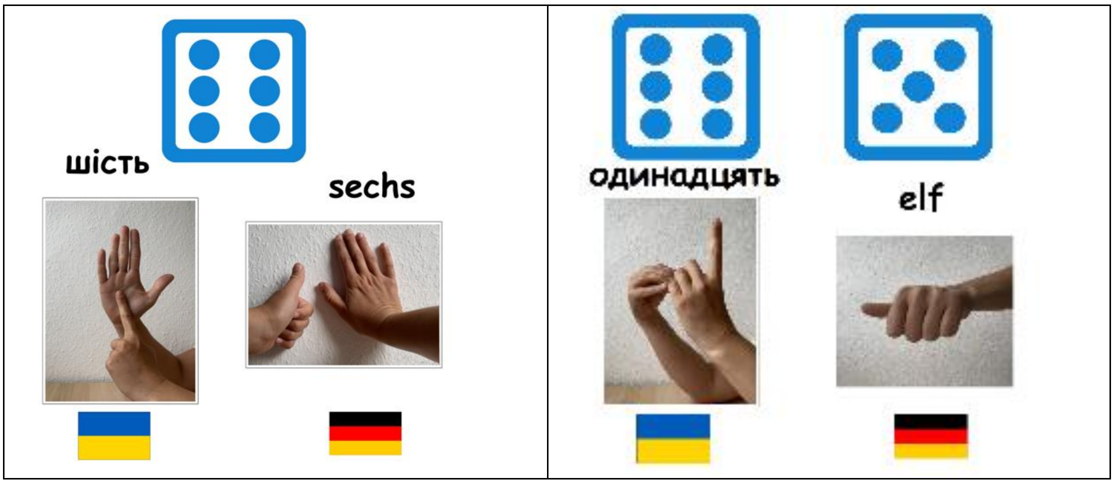


Figure 1: Examples for numbers in German and Ukrainian Sign Languages (Copy right: Maike Beyer)

These examples already indicate the symbolic nature of the signed numbers and the complexity of the differences which can affect school children's understanding by transitioning between languages. While in the signed number "six" the number of fingers corresponds to the cardinality of the number, in the Ukrainian variant of the signed number "eleven" the five fingers of the left hand and the four fingers of the right hand of the person signing represent the ten. In Berlin's version 11 is represented with the help of movements (see Figure 10) as you can see later exemplified by Olga Pollex (Frau TAUBe). However, the creation of signed numbers, hand configurations and movements are not arbitrary, but follows not only linguistical constraints but also systematical and logical mathematical rules (cf. Werner et al., 2019).

Leybaert and Van Cutsem (2002) examined the accuracy and use of the number sequences in hearing children aged 3 years and 4 months to 5 years and 8 months and in deaf children aged 4 years and 6 years and 2 months. Three tasks were used: abstract counting, counting objects and forming sets with a specific cardinality. Deaf children showed age-related delays in their knowledge of the number sequence. The deaf children's errors were not arbitrary and could be attributed to the rules of sign language. They found that deaf children made more errors when they counted up to number 6. This is the first time where the additive rule applies in Belgian Sign Language. Remarkably, their performance in counting objects and forming quantities of a certain cardinality was similar to that of hearing children, although hearing children had a longer number sequence. This suggests that deaf children are better at counting and number representations than their knowledge of the number sequence would suggest.

In a comparative study on counting skills in German Sign Language (DGS) and German of six-year-old deaf DGS signing and hearing German speaking children, it was shown that the use of number signs has a particular influence on the naming of successor numbers. The larger the number whose successor is to be named, the more consistently deaf children performed. In hearing children, on the other hand, the solution rates with German number words decrease. The special number sign structure in DGS can therefore support the development of counting skills (Werner & Hänel-Faulhaber, 2024).

In an earlier study, Nunes and Moreno (1998) investigated the use of calculation algorithms in British Sign Language (BSL) by deaf children. The errors observed in this study by deaf children were systematic errors and not random, incorrect counts. The deaf children's errors could be directly linked to the structure of the counting system and the algorithm used, just as the errors in written arithmetic were linked to the understanding of place value and the mechanics of the written algorithm. These results illustrate the effects of a sign system on mathematical reasoning. They show how sign numbers influence the arithmetic process of deaf children (Nunes & Moreno, 1998). Signed algorithms are complex mathematical phenomena. To understand the dimensions of the complexity of signed algorithms, we recommend the work on signed algorithms in Finnish Sign Language by Rainò et. al. (2018).

To summarize our report on recent findings we would like to note that the positive effect of Sign Languages on learning mathematics by deaf chidren has been empirically proven. Sign Languages support short- and long-term memory by allowing additional sensory (kinematic) experiences of numbers, number spaces and algorithms. Signed algorithms also provide an additional symbolic tool for solving mathematical problems and are relevant in the context of cultural affiliation to national Sign Language communities. They are complex phenomena that pose a challenge for interpreting (cf. Rainò et. al., 2018). Their importance for the successful learning of mathematics is difficult to overestimate.

In addition to targeted language support in the context of mathematical teaching, it is important to specifically promote the visual-imaginative thinking of deaf children. Following Rosanova (1978) and Yashkova (1988), we use this term to describe the ability to interpret mathematical visualizations or models and to mediate them supported by languages, to use them to solve mathematical problems and to develop them independently using signs. In contrast to Rosanova (1978) and Yashkova (1988), we want to place a stronger focus on sign languages.

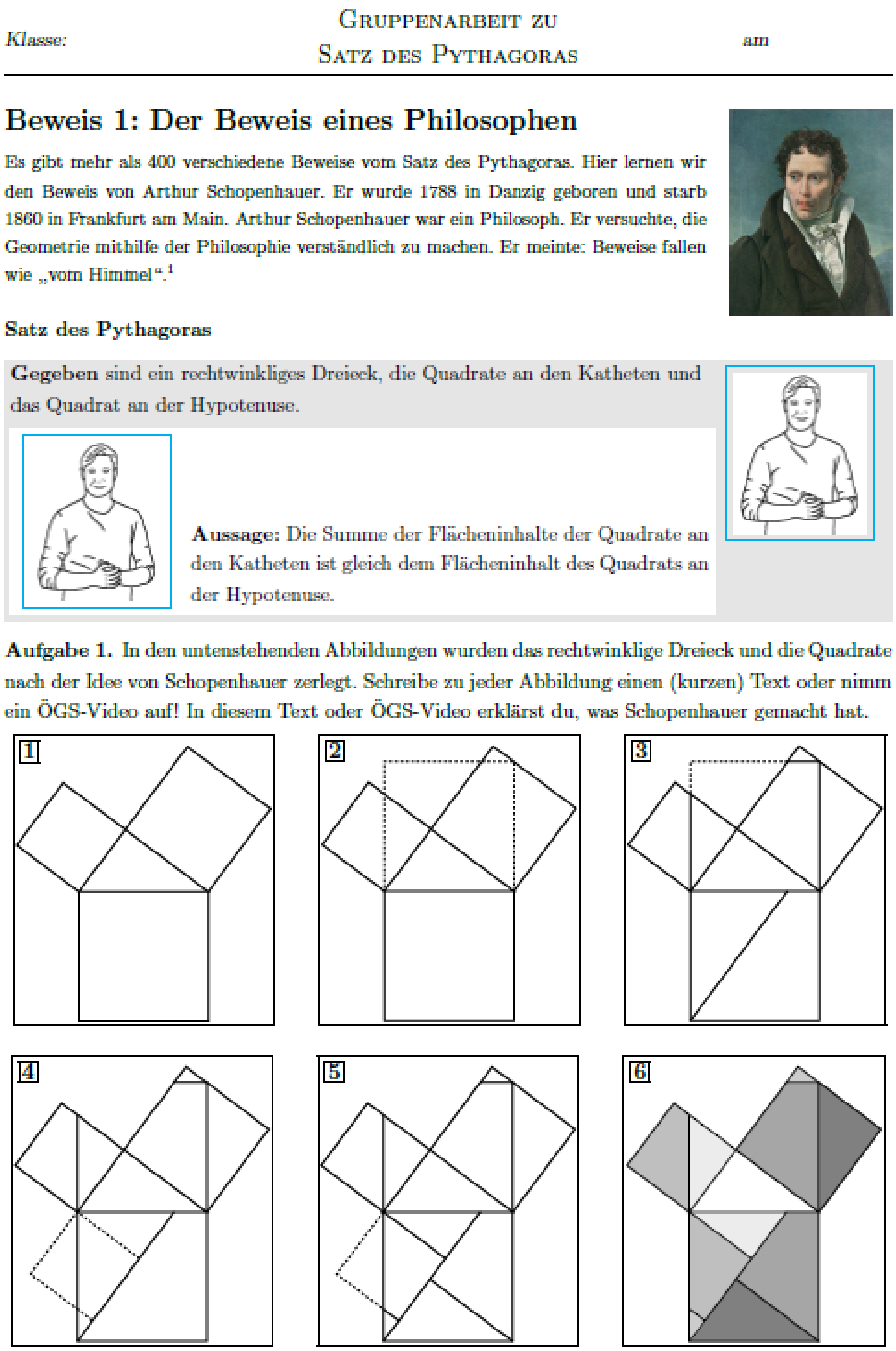
Selected Intervention Studies

In the following, we will present some examples of selected intervention studies with deaf children. Here, too, we make no claim to completeness, but use presented studies to mark the need for development of concepts and materials for mathematical teaching which implement sign languages. These are primarily the intervention study by Nunes and Moreno (2002), its further development by Wille (2018, 2019) and the intervention study by Angeloni and Wille (2022).

Nunes and Moreno (2002) developed an intervention program to promote the numeracy skills of deaf children. They compared 23 deaf learners who participated in the project with a baseline group consisting of 65 deaf learners who had attended the same schools in the previous year. The participating learners were tested before and after the intervention with the Nelson Age-Appropriate Mathematics Achievement Test. The intervention was delivered by teachers during the time normally allocated for mathematics lessons. The learners who took part in the intervention study by Nunes and Moreno (2002) did not differ from the control group in the pre-test but performed significantly better in the post-test. Nunes and Moreno (2002) came to the conclusion that the intervention program effectively promoted the performance of deaf learners in arithmetic. Nunes (2004) assumes that deaf children's strengths lie in the processing of spatial-visual information. Based on this assumption, she proposes teaching materials for four basic arithmetic operations that contain graphic representations and questions in written English.

One of the strengths of the intervention study by Nunes and Moreno (2002) is the fact that the tasks, diagrams and task texts are made available to the teachers. However, sign language representations of numbers or other technical signs are not integrated into the worksheets presented. In this sense, the studies by Wille (2018, 2019) based on Nunes and Moreno (2002) have particular theoretical and practical relevance for sign language mathematics lessons. In these studies, the concept of Nunes (2004) was further developed and sign language explanations were specifically taken into account and documented as videos. The studies were tested in the context of specific learning groups at an Austrian school in which deaf learners were included. The development work was theoretically located in the context of semiotics and taking into account the work of Kutscher (2010). The materials developed can be found here: http://www.annikawille.de/mathe_in_oegs/mathe_in_oegs.html

In a study by Angeloni and Wille (2022), multi-modal learning environments with videos, worksheets and comics on the Pythagorean Theorem were developed in cooperation with Christian Hausch and tested in a group of deaf learners (Angeloni & Wille, 2022). The materials can be used in bimodal bilingual lessons and contain both sign language and written language explanations and mathematical problems. One of the most important results of the work was the finding that the materials offered in Austrian Sign Language were better accepted and processed by the participants than materials in written German. At the same time, Angeloni and Wille (2022) note a very high workload in the development of teaching materials that include Sign Languages. A particular strength of the study is the openness to learners' signed explanations and variations. For example, in one task, learners are encouraged to complete the parts of the proof of the Pythagorean theorem presented as a sequence of pictures using sign language or written language and to record them on video (see Figure 2). Here we can see an example of how the theoretical proposals of Rosanova (1978) and Yashkova (1988) can be used didactically and methodically in the classroom using modern media. It is worth emphasizing that in the study not only one, but several proofs of a sentence were thematized.



Klasse: | Gruppenarbeit zu Satz des Pythagoras | am

Beweis 1: Der Beweis eines Philosophen

Es gibt mehr als 400 verschiedene Beweise vom Satz des Pythagoras. Hier lernen wir den Beweis von Arthur Schopenhauer. Er wurde 1788 in Danzig geboren und starb 1860 in Frankfurt am Main. Arthur Schopenhauer war ein Philosoph. Er versuchte, die Geometrie mithilfe der Philosophie verständlich zu machen. Er meinte: Beweise fallen wie „vom Himmel“.[1]

Satz des Pythagoras

Gegeben sind ein rechtwinkliges Dreieck, die Quadrate an den Katheten und das Quadrat an der Hypotenuse.

Aussage: Die Summe der Flächeninhalte der Quadrate an den Katheten ist gleich dem Flächeninhalt des Quadrats an der Hypotenuse.

Aufgabe 1. In den untenstehenden Abbildungen wurden das rechtwinklige Dreieck und die Quadrate nach der Idee von Schopenhauer zerlegt. Schreibe zu jeder Abbildung einen (kurzen) Text oder nimm ein ÖGS-Video auf! In diesem Text oder ÖGS-Video erklärst du, was Schopenhauer gemacht hat.

Figure 2: Series of pictures on the Pythagorean theorem (Angeloni & Wille 2022)

One of the interesting results of the study was the observation of the difficulties deaf learners had when using geometric terms. For example, they were unable to recognize the right angle in the sign when its position was changed, and the vertices of the angle were no longer parallel or perpendicular to the ground. In Olga

Pollex's suggestions, we will see how these difficulties could be dealt with in the classroom or how they could be prevented by making greater use of enactive teaching aids and productive mathematical signs, which Angeloni and Wille (2022) refer to as classifiers.

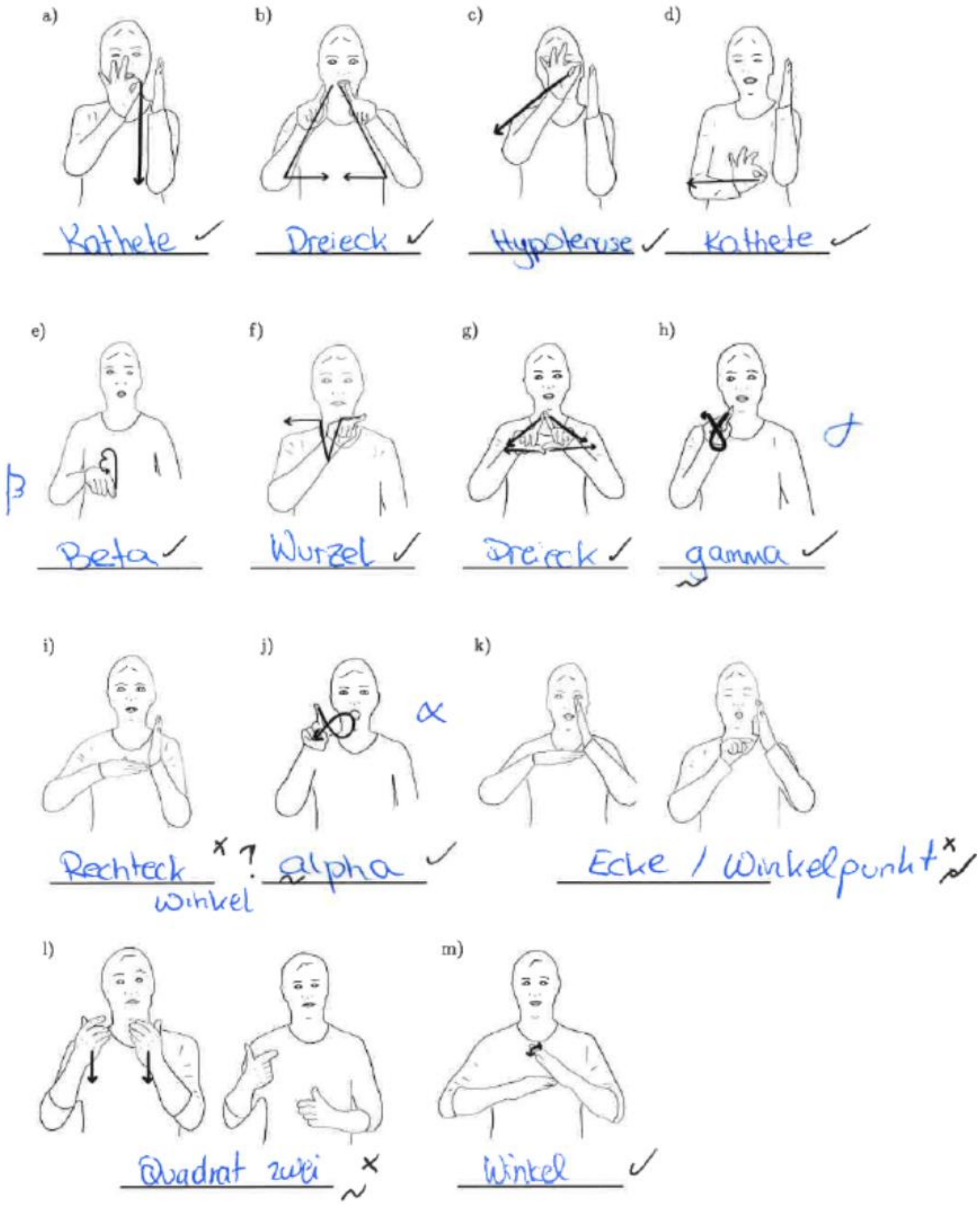


Figure 3: Vocabulary used in the intervention study by Angeloni & Wille 2022

One of the important results of the study is the decision to focus more on Sign Languages in future studies and to largely dispense with the written language dimension. The documented feedback and learning progress of learners in the works by Wille (2018, 2019) and Angeloni and Wille (2022) show examples of how Sign Language teaching materials can be used in bimodal bilingual lessons. However, they also show that there is a need for more teaching materials and concepts that specifically take Sign Languages into account and link them with other teaching media. In the next step, we will give some examples of teaching materials that have been developed for deaf children and then turn to some innovative methods and teaching materials.

Examples from Practice and Innovative Teaching Methods

Before we bring current and innovative examples from the practice of teaching mathematics, we will first turn to history and present examples from the teaching methodology of Fleri (1835) as well as excerpts from the documents in which tasks and methods are presented that were already used in the teaching of geometry in Sign Languages in the 19th century (Tabak 2014). We will then look at current textbooks for deaf children and then present examples from the lessons of Olga Pollex, head of the specialist seminar for "Hearing and Communication" and teacher of mathematics.

Lessons from history

The idea of using signs and Sign Language teaching materials in the classroom has a long tradition. Fleri (1835), for example, gave a didactic and methodical introduction to sign numbers and described some important mathematical signs in one of the first sign lexicons. Signed mathematics is first introduced with beans or sticks, then with the help of line drawings. It is interesting to note that the sticks are structured in such a way that the base 5 can be quickly grasped visually. Fleri (1835) based his considerations on his observations of arithmetic algorithms that deaf people use among themselves (see Figure 4).

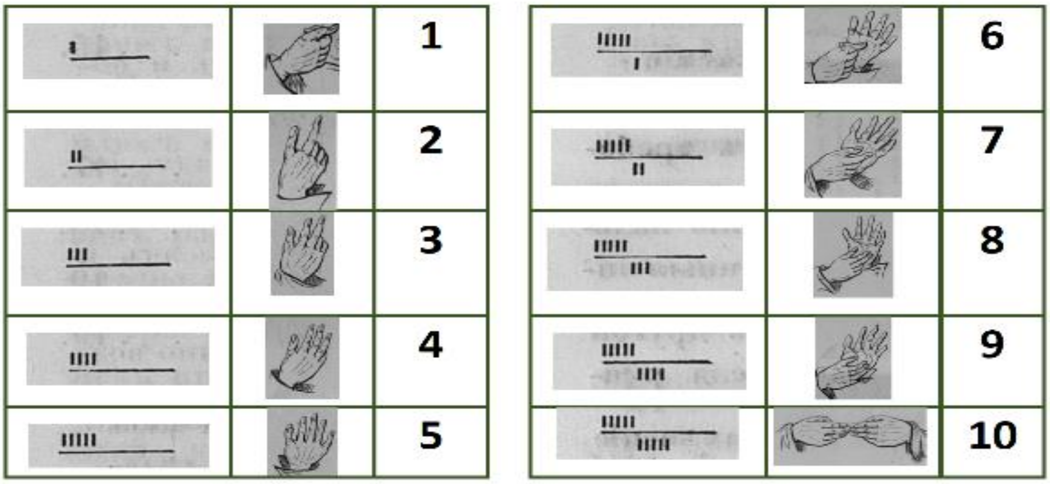


Figure 4: Introduction of numbers from 1 to 10 (source: Fleri 1835)

Tabak (2014) looked at the mathematical terminology used to teach mathematics in Gallaudet University or its predecessor institution Columbia. He has found old documents describing examination tasks and teaching methods used for geometry, which state: “[Geometric] demonstrations are occasionally made in writing, but the usual course is for the student to draw a diagram, and to give the proof by means of signs and the manual alphabet, pointing out each angle [and] line…as it is needed in the argument” (Nineteenth Annual Report of the Columbia Institution, 1876, p. 5-6). This method of teaching clearly required that both faculty instructors and students develop a “mathematical extension” of American Sign Language (ASL) sufficient to express Euclidean geometry (see Figure 5).

II.—*Mathematics.*

Geometry.—The Freshman Class, in its first two terms, is taken through the first eight books of Loomis's Geometry. Demonstrations are occasionally made in writing, but the usual course is for the student to draw a diagram, and to give the proof by means of signs and the manual alphabet, pointing out each angle, line, &c., as it is needed in the argument. Several weeks are devoted to the demonstration of theorems not demonstrated in the text-book. In the examination upon the first six books, only the numbers of the book and of the proposition are given to the student; in the remaining books, the theorems are given.

Geometry is completed in the third term of the Freshman year. The spherical black-board is used in the discussion of spherical triangles.

Algebra is resumed, and the subjects of ratio and proportion, progressions, permutations, the binomial theorem, series, and the general theory of equations are studied. The recitations are chiefly written, and the students elucidate their work by means of the sign-language and manual English.

Conic Sections are studied in the first term of the Sophomore year. Sixty-five propositions are demonstrated, numerical exercises are solved, and a few theorems are assigned for original work.

Plane Trigonometry and Logarithms are studied in the second term. The text-book is well supplied with exercises, which are used to test the knowledge and comprehension of the class.

Spherical Trigonometry is studied in the third term. The method is the same as in plane trigonometry. In some classes the work has been abridged, and the time devoted to Mensuration and Surveying.

Mechanics.—All the elementary propositions of mechanics are mathematically demonstrated, and illustrated by numerous practical examples. The Juniors study mechanics for one term.

The principles of mechanics as applied to astronomy are studied by the Juniors in the third term, as noted under the head of Natural Science.

Figure 5: Undergraduate courses of mathematics at Columbia Institution (1876)

Given this description of how geometry was learned and the test questions found in the older documents, it is possible that mathematical ASL, at least with respect to Euclidean geometry, was as well developed in 1876 as it is today, perhaps even better.

Unfortunately, we don*t have information how the concrete technical signs were documented at Columbia Institution. We only could find the written tasks and can see what mathematical content was embedded in Sign Language at that time. Tabak (2014) gives some examples of exam questions. Here are two examples from geometry:

- ***"Construct a plane triangle, having [been] given the perimeter of the angles of the triangle." (p. 32)***
- ***Conic Sections: "Prove that perpendiculars drawn from the foci upon a tangent to the ellipse meet the tangent in the circumference of a circle whose diameter is the major axis." (p. 32)***

Examples of special textbooks

On the basis of empirical studies and interventions, a series of methodical handbooks and special mathematics textbooks for deaf preschool and primary school children have been and are being developed and tested in Russia and Georgia, for example. Sign language representations of numbers were included in the textbooks (see for example Suchova 2002). They are presented alongside numbers in Arabic notation and in pictures. For example, there is a picture of ten frogs. The frogs are grouped in such a way that the children can recognize patterns (3+3+3 or 3x3) and 9 as a number they already know. Another frog which is placed outside of the group of nine increases the number up to 10. This gives preschool children the opportunity to get to know the number 10 in the context of visual patterns, as successor of the number nine and as signed number ten which is known at least to those of them whose parents are signing. The signed presentation is at the beginning of introduction to number ten. In the following, the children are offered the tasks in pictures or sequences of pictures. The tasks are deliberately chosen so that the correct result is not always 10. This means that the children have to think, count, calculate and use visuals to solve the problems by themselves. The pictures are linked to contexts that the children may be familiar with from their everyday lives.

Newer Russian textbooks like "Mathematics 2" (2023) or those which are aimed at older deaf children, contain pictorial-schematic representations of text problems as well as adapted simplified texts in Russian. Similar textbooks for deaf children were developed for deaf children in Berlin in the 1970s and tested in special schools for deaf learners.

As there is a lack of materials with explanations in Sign Languages, which are connected to visual aids and mathematical representations, teachers are developing materials and methods that can fill these gaps to meet the needs of their heterogeneous learning groups of deaf school children. We have thus arrived at a point in the didactics of mathematics where school practice is ahead of didactical research, and therefore we are turning not only to scientific literature but pose our questions to Olga Pollex, whose professional expertise and experience lies in the intersection of didactics of sign languages and mathematics.

Olga Pollex is an experienced teacher trained in mathematics and special education and is also the head of a seminar for young teachers for the deaf. In the following, we quote written statements from correspondence and conversations with Olga Pollex. In the context of ethnographic research methodology, we decided to present original excerpts from Olga Pollex's arguments and recommendationsfor teaching. These are enriched with concrete teaching examples which are intended to serve as a source of inspiration for teachers in practice and a treasure trove of ideas for Design Based Research in mathematical education of the deaf learners.

Innovative teaching methods using selected examples

As a specialist with theoretical and practical knowledge, Olga Pollex is firmly convinced that “wordless”, “language-free” and even “language-poor” teaching materials are not sufficient to teach deaf children in mathematics in the long term:

“To understand and explain processes in math, you need language. I often find that language is underestimated and omitted in mathematics lessons. School children are then able to understand certain task formats through frequent practice, but they are unable to understand the processes behind them. For this reason, there are often problems with understanding and explaining their own calculation methods and algorithms.

The solutions are often developed by copying the calculation methods without understanding why this is done in a certain way. Mathematical tasks which foster problem-solving, transferring one's own knowledge and transfer tasks are important parts of math lessons. I often observe that math lessons for deaf children focus more on mechanical arithmetic, where aspects of the mathematical language are underestimated.

The Sign Language skills of teachers and learners certainly play a role here. In order to avoid Sign Language, mathematics lessons are often focused on automating skills (reproducing exercises). Language is needed to recognize mathematical structures, relationships, to link knowledge and skills and to transfer these to unfamiliar problems.

To illustrate her thoughts on the topic of Sign Languages in mathematics lessons, Olga Pollex applies the model used in school-oriented STEM-project called SINUS. By doing this she analyses challenges teachers and school children have when it comes to the use of German Sign Language (DGS) in mathematical lessons.

In the Figure 6 we will summarize the levels of used language translated into English. It is important to note that the levels are interlinked and interrelated. Positive developments at one level can lead to progress at other levels. Conversely, difficulties and developmental delays at basis level like the “everyday language”, for example, can affect the development at other levels and mathematical development in general.

Mathematical symbolic	***¾***
Mathematical terminology	***fractions***
“Mediating” language of instruction	***part of the whole***
Everyday language	***slice of pizza***

Figure 6: The levels of used language (Retrieved from: https://www.schulentwicklung.nrw.de/sinus/upload/Publikationen/Ma5-10_38_Heft_04-05.pdf)

The difficulties that we teachers of deaf sign language-oriented children encounter in the classroom are based on the sign language skills of the teachers and the learners. I would now like to discuss this separately.

The language competence of the learners:

Deaf learners with language deprivation, who had no possibility to learn Sign Languages at home already have difficulty with “everyday language” in Sign Language when they come to school. When they come to school, language work generally has to be done. “Mediating” language of mathematical instruction is a level which can be too high for these children.

However, deaf learners who have appropriate Sign Language skills according to their age from home are capable of everyday Sign Languages. I could very often experience that “mediating” language of mathematical instruction in German Sign Language is accessible for them.

The language skills of the teachers:

It is of course a challenge for many teachers to teach the subject of mathematics in Sign Languages. If there are sign-language-oriented children in the group, I think it is important that it is taught in German Sign Language and that not only spoken language, which is supported by single signs, is used. It is also important that many productive signs and the signing space are used in the subject like mathematics.

Suggestions for promoting and compensating for sign language skills in Mathematics lessons

My suggestion for teachers and also for the learners is, on the one hand, greater consideration and interconnection of the ***enactive****,* ***iconic*** *and* ***symbolic*** *levels (according to Bruner) and increased use of* ***productive signing****, which is directly linked to enactive actions and iconic representations.*

Teachers often ask me about technical terms. These signs are very important. However, if we look at the model in Figure 6, mathematical terminology comes in on the third level after everyday language and "mediated" language of instruction.

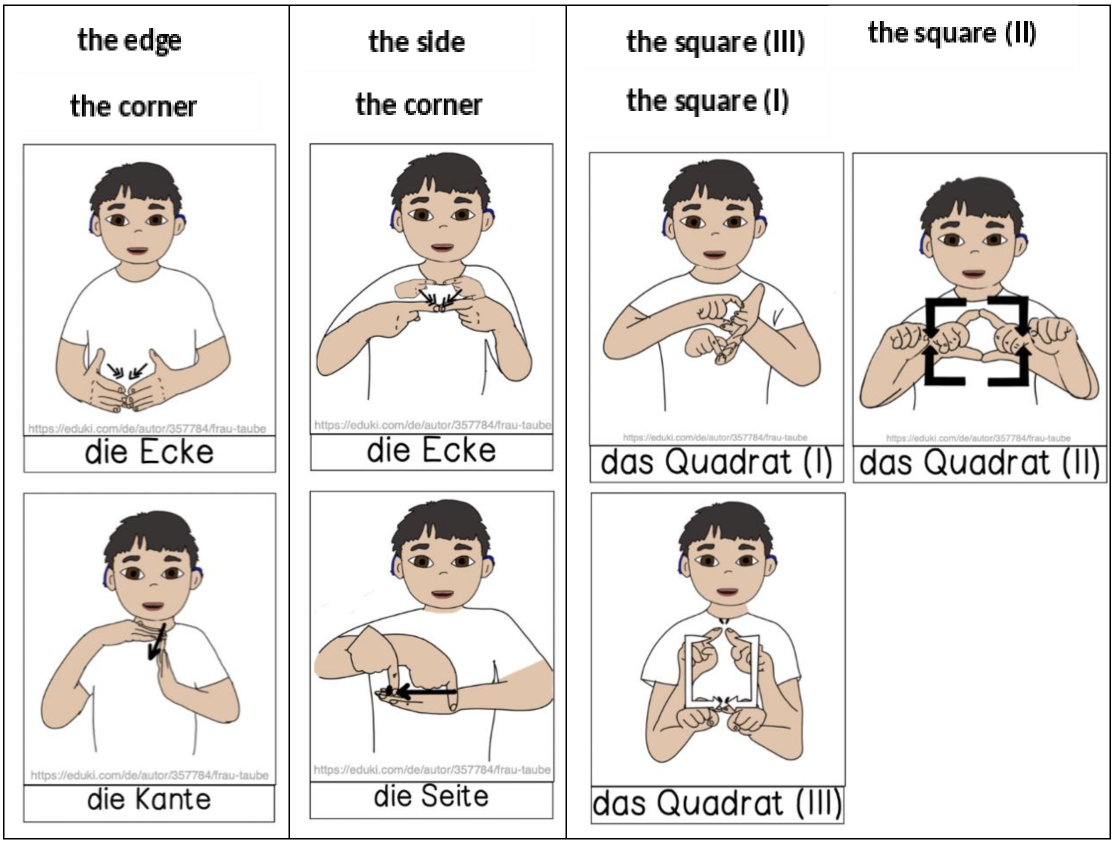
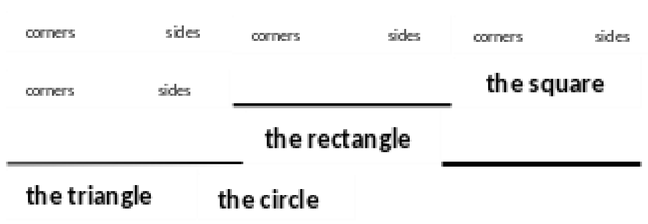


Figure 7: Sides, edges, squares and corners for 2-D and 3-D figures (Copy right: Frau TAUBe)

To enable access to mathematics, increased use of productive gestures and the gesture space is important. Only then could special signs as signed mathematical terminology be introduced and then represented using the symbolic language of mathematics.

I often observe that teaching is done in German Sign Language, but in a linear way. This means that signs are simply strung together without using the space and productive signs. We should try to get away from arranging the signs in a one-dimensional sequence and really work more spatially with signs. It brings many advantages in math lessons. The use of space is not easy to represent on 2D worksheets, but I try to transfer the multidimensional and dynamical structures of the signs to design of the worksheets. Here is an example of productive signs (Figure 7).

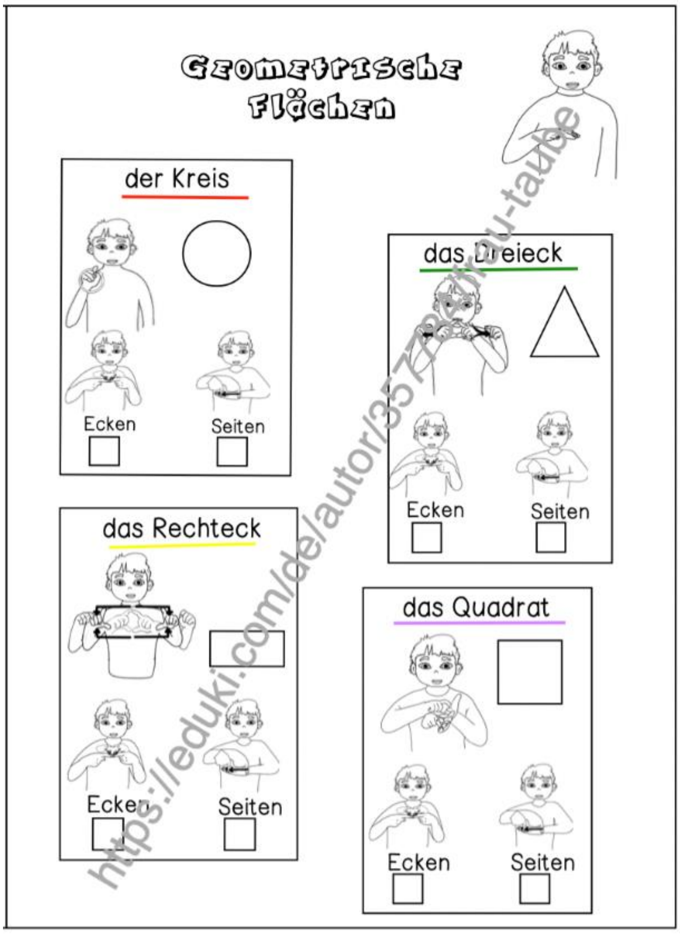
For specialist signs, such as the sign that corresponds to the mathematical term "square", it is important to show several variants and to be open for variations of the sign the learner can suggest. After different signs are introduced and discussed in the classroom the learners and teachers can decide together which sign should be used as a fixed variant in the particular classroom (Figure 7).



It can also be helpful to give all the central terms together at the same time so that the learners can see similarities and differences at a glance. The example in the illustration on the left is about different geometric surfaces and the number of their sides and corners. In this way, the terms are not only embedded in context, but also in a small exercise. The terms can thus be immediately incorporated into the linguistic repertoire of the teachers and the learners.

What is also special about this worksheet? Geometric visualizations, gestures and words are linked directly in the learner's field of vision, which is in line with the theory of Rosanova (1978) and Yashkova (1988) and our extension of their model.

Geometric surfaces



It can also be helpful to give all the central terms together at the same time so that the learners can see similarities and differences at a glance. The example in the illustration on the left is ab out different geometric surfac es and the number of their sides and corners. In this way, the terms are not only embedded in context, but also in a small exercise. The terms can thus be immediately incorporated into the linguistic repertoire of the teachers and the learners.

What is also special about this worksheet?
Geometric visualizations, gestures and words are linked directly in the learner 's field of vision, which is in line with the theory of Rosanova (1978) and Yashkova (1988) and our extension of their model.

Figure 8: Corners and sides for 2-D and 3-D figures

Pythagoras Theorem: Example for use of productive signs

To illustrate how productive gestures can be used in lessons, I would like to look at an example from my lessons, namely the introduction of the Pythagorean theorem by linking actions and gestures. The illustrations from the Sign2MINT database (Barth et. al., 2022) *show technical language descriptions of the theorem, which also illustrate equality of area in the conventionalized version with the help of the corresponding hand shapes and execution points.*

Figure 9: Specialized gesture for PYTHAGORAS THEOREM (c) Sign2MINT@Max Planck Foundation

Looking back to my introduction of Pythagoras Theorem I suggest to start with actions or working on enactive level. I gave my learners the possibility to experience the equality of the sum of the areas by experimenting with paper figures. That is how they can figure out with help of concrete examples from paper that the squares of the cathets and the square of the hypotenuse have the same values in these particular cases. They could cut and cover the areas using a few examples. The situation experienced on the enactive level should then be described by the learners in their own sign language. Through the narrative, which included productive signing, they were also able to understand the law again and transfer it to other factual tasks that could be solved with the help of Pythagoras' theorem.

By introducing the learners to the Pythagoras Theorem through acting and signing, they were later able to understand technical signs and the proposition of the Theorem formulated in the technical signs.

It is not only geometry that offers the possibility of linking productive signs with actions. Here I would like to look at a few more examples / topics to discuss language:

Numbers over 10

It is possible to introduce to sign language and its mathematical symbolics by using numbers. Up to numbers 10 the signs for numbers are concrete. The number of fingers in the signed number from 1 to 10 represents the number. The number 11 and bigger use hand movements as symbolic parts of the signs to represent tens.

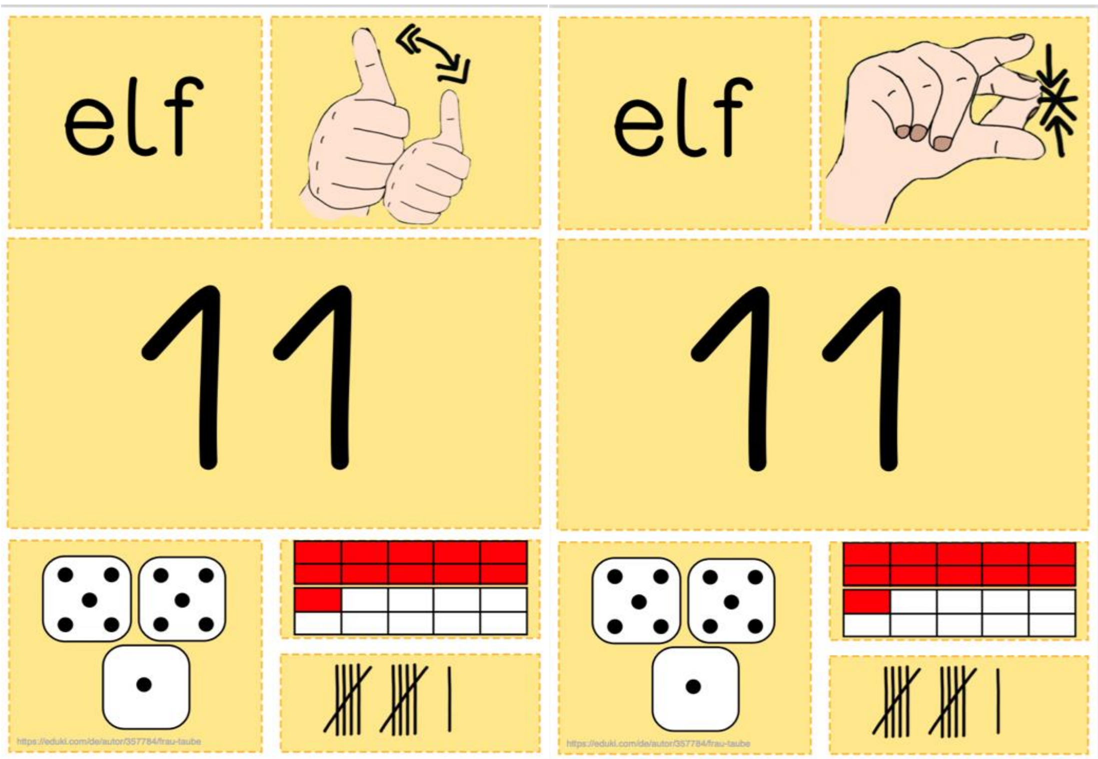


Figure 10: Two variants for signed eleven used in Berlin (Copy right: Frau TAUBe)

There are dialects in Germany. The Berlin version of the numbers "11" and "12" is mathematically confusing. The number 11 is tapped with the thumb and index finger and is often confused with 12. Number 12, where the thumb taps the index and middle finger. It is therefore confused with 13. And here, as a math teacher, you are faced with a decision. Do I follow linguistics and show the Berlin signs, as I teach in Berlin? Or do I better adopt the version where the numbers 11 and 12 are "shaken" just like *the numbers 13-19? I always show both versions and then always take the "more complicated" 11 and 12. To school children with language deprivation syndrome or learning difficulties I show other versions. I often leave it up to the children which version they can use. I also always tell them that the numbers are signed differently in Germany and other countries in order to create flexibility.*

When dealing with numbers, it is important to me to incorporate aspects of fostering abilities to communicate in German Sign Language into math lessons. For me it is important to pay attention to the correct use of parameters when presenting technical signs and numbers as shown in the example in Figure 11. Correct execution is circled in green, while incorrect execution of numbers is shown in the red circles.

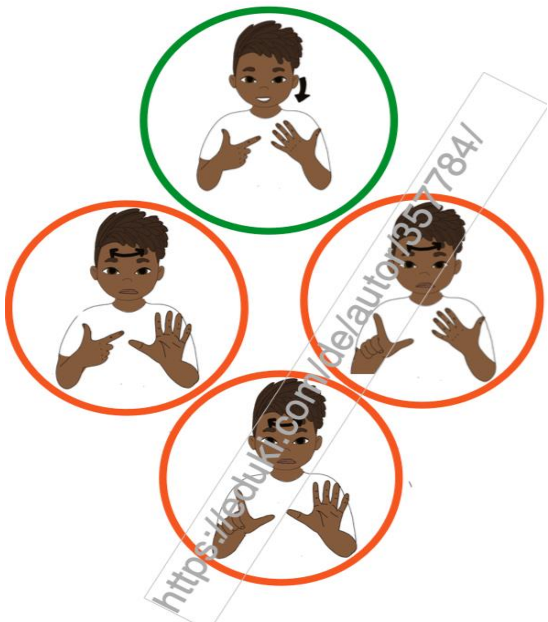


Figure 11: Lexically right way to sign a 7 in DGS (green) and typical mistakes (red), (Copy right: Frau TAUBe)

Arithmetic over 10

To give an example for first signed calculation I would like to refer to the interesting experience with doubling singed algorithms which can help with arithmetic for beginners. You sign both numbers in the air. Both fives are bundled together to form a 10 and the remaining numbers are added.

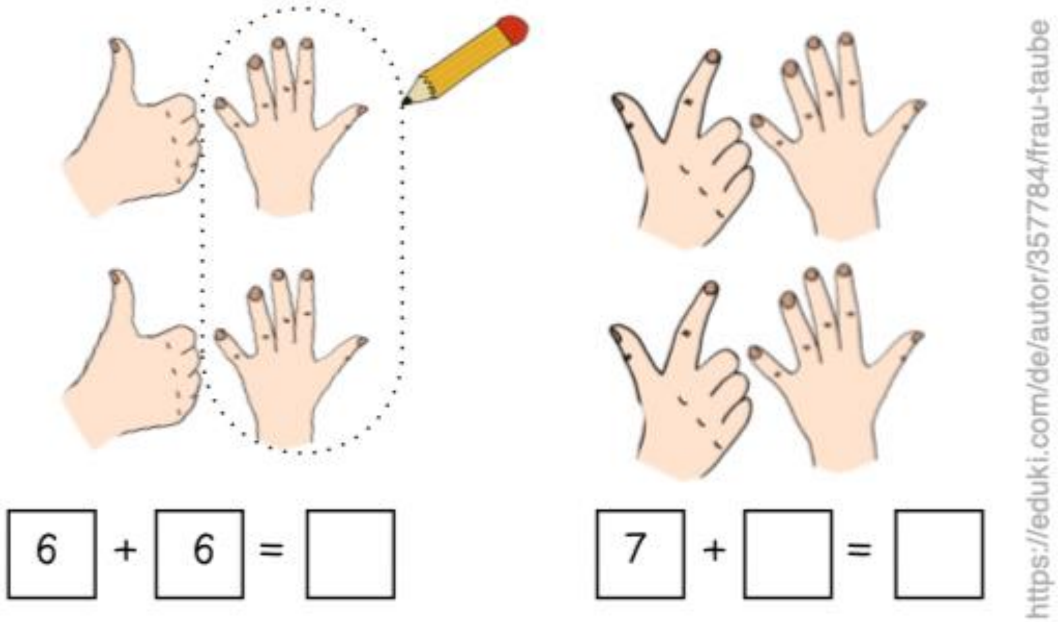


Figure 12: Signed doubling algorithm (Copy right: Frau TAUBe)

However, it becomes difficult, for example, when calculating 9+5. Ten fingers are no longer enough. So, you have to operate with mental number concepts in your head without using hands. I have observed that learners who mouthed numbers were able to fix number names in this way and to continue counting in their heads supported by mouthing numbers leaning on symbolical aspects of number line. For learners who were signed without mouthing of number names found it difficult to continue. But these are just my observations, and I can't say that this is the general rule. In any case, it's a frequent observation, that is why I also integrate practicing of mouthing of numbers in German into my lessons.

Examples from combinatorics

Just to give an idea what could be important by teaching combinatorics I would like to refer to my experience with teaching lessons in combinatorics called "Permutation, variation and combination". The structure of my teaching units is described in the Figure 13.

Aufbau der Unterrichtseinheit:

	Mathematischer Inhalt	Gestaltung der Stunde
1. Stunde	Variation mit Wiederholung	Einführung in die Geschichte (Fridolin will den Kuss) Aufgabe: Wir knacken das Schloss Tätigkeit: 2 von 3 Symbolen in einer bestimmten Reihenfolge im Schloss richtig einstellen
2.	Variation ohne Wiederholung	Aufgabe: Unterschiedliche Legotürme bauen. Tätigkeit: Aus 3 Bausteinen mit je einer Farbe einen Legoturm aus 2 Teilen bauen.
3.	Kombination ohne Wiederholung Kombination mit Wiederholung	Aufgabe: Möglichst viele Eiskombinationen für die Prinzessin anbieten. Tätigkeit: 2 Sorten aus 3 Sorten Eiskugeln wählen. Tätigkeit: 2 unterschiedliche Sorten aus 3 Sorten Eiskugeln wählen.
4.	Permutation ohne Wiederholung	Aufgabe: Für Fridolin die Krone auf möglichst viele unterschiedliche Arten gestalten. Tätigkeit: 3 Glitzersteine in bestimmter Reihenfolge aus 3 Sorten wählen und auf die Krone kleben.
5.	Variation ohne Wiederholung Permutation ohne Wiederholung	Aufgabe: Kleid der Prinzessin mit den Knöpfen möglichst unterschiedlich ausschmücken. Zwei Knöpfe in einer bestimmten Reihenfolge aus 3 Sorten wählen. Drei Knöpfe in einer bestimmten Reihenfolge aus 3 Sorten anbringen.
6.	Kreuzprodukt	Aufgabe: Kleidung für Fridolin aus 2 Hosen und 3 Taschen auswählen. Zusatzaufgabe auf enaktiver Ebene: Kombinationen aus Schal und Mützen entwickeln

Figure 13: Structure of the teaching unit "Permutation, variation and combination". (Copy right: Frau TAUBe)

Structure of the lessons:

	Mathematical content	Organization of the lesson
Lesson 1	**Variation with repeat**	Introduction to History (Fridolin wants the kiss) **Task:** We pick the lock **Activity:** Set 2 of 3 symbols correctly in a certain order in the lock
Lesson 2	**Variation without repeat**	**Task:** Build different Lego towers **Activity:** build a Lego tower made of 2 pieces from 3 building blocks with one colour each
Lesson 3	**Combination without repeat** **Combination with repeat**	**Task:** Offer as many ice cream combinations as possible for the princess **Activity:** Choose 2 types from 3 types of ice cream scoops **Activity:** Choose 2 different types from 3 types of ice cream scoops
Lesson 4	**Permutation without repeat**	**Task:** design the crown for Fridolin in as many different ways as possible **Activity:** Choose 3 rhinestones in a certain order from 3 varieties and glue them to the crown
Lesson 5	**Variation without repeat** **Permutation without repeat**	**Task:** Decorate the princess's dress with the buttons as differently as possible **Activity:** Choose 2 buttons in a specific order from 3 varieties **Activity:** Attach 3 buttons in a specific order of 3 varieties
Lesson 6	**Cross**	**Task:** Choose clothing for Fridolin from 2 pants and 3 pockets **Additional task at the enactive level:** Developing combinations of scarf and hats

Figure 13b: Translation of figure 13

Tasks and problems from combinatorics require linguistic understanding. I was unsure whether my group of young learners would understand the tasks. Learners with high abilities in German Sign Language understood very quickly the problems and their fine variations. To involve learners which have certain difficulties with German Sign Language I had to demonstrate the meaning of the tasks by actions and derived productive signs from these actions. I then realized that almost all of the pupils were able to understand the tasks and problems well and were able to work on them mostly independently and in group work. At the end, they were always able to explain to me how they found the combinations which were described in the problems. Figure 14 gives an example of explanation in German Sign Language documented partly on the work sheets.

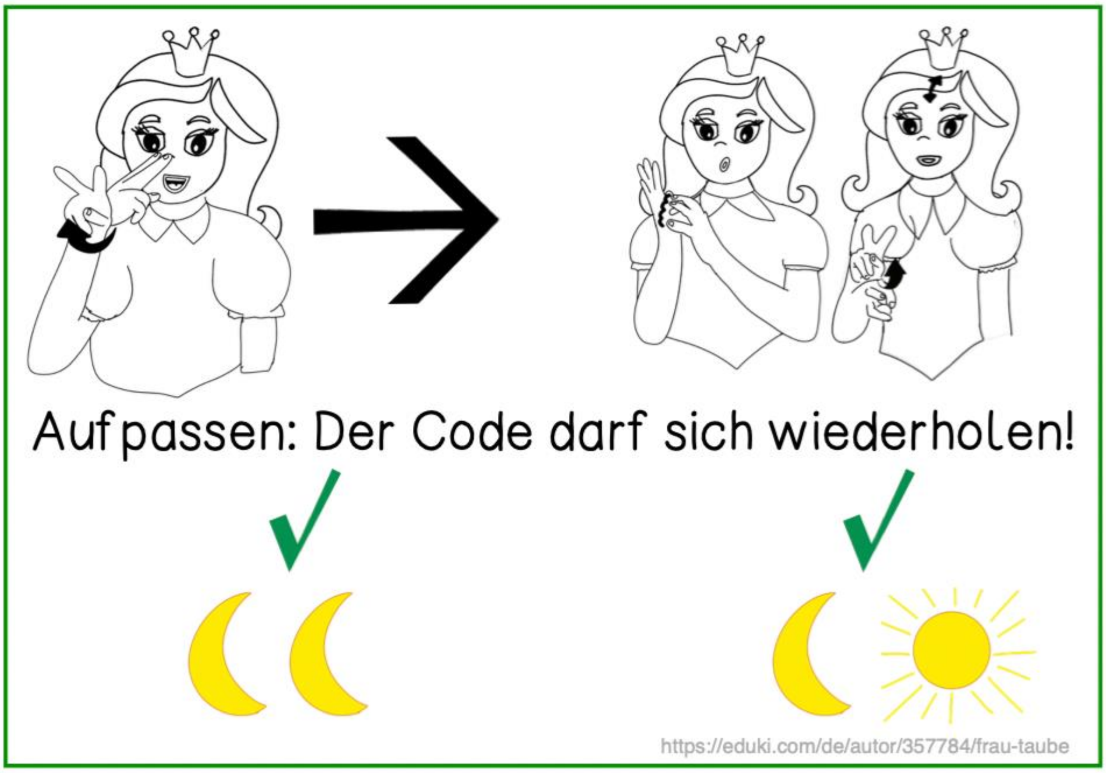


Figure 14: Worsheet: Watch out! The code may be repeated. (Copy right: Frau TAUBe)

As we already have seen in the examples for combinatorics problems which are formulated as written texts but also signed story problems can be very challenging for learners with language deprivation. I would like to end with some observations of story problems formulated as written text or signed.

Story problems

Talking about story problems, I would like to refer to the seminar conducted by Viktor Werner which leaned on his theoretical and practical expertise. In this seminar teachers were given the task of translating the text of mathematical problems in written German into German Sign Language. By trying to translate the problems teachers noticed that their signed version very often included a solution or at least parts of it. I have therefore come to the following conclusions for myself and teacher students I am working with: ***"When translating, make sure that challenging aspects don't' disappear from the problem. In addition, develop your own tasks and problems directly in German Sign Language, because the linguistic structure of signed languages is different from the spoken and written language. Problems which are originally conducted in sign languages use signing more naturally. I am dreaming of my own task pool in German Sign Language (QR with access to sign language videos). Maybe it will come about. The signed problems in the Advent calendars are the first steps."***

At the same time, it is important that we also provide access to written language and practice reading skills. You always have to decide whether you only want to discuss mathematical problems or whether you also want to practice using German. For us, it's always a double task. Dealing with signed problems or text problems is a huge topic in itself. I hope that will address this didactically challenging topic separately at another point in the future.

The bottom line is this: Mathematics and language are closely linked. And in order to give deaf children access to mathematics, it is necessary to provide them with linguistic tools.

Closing the practical considerations and examples provided by Olga Pollex we recommend her teaching materials, which can be found here: Frau TAUBe | Unterrichtsmaterialien bei eduki.com

We would like to follow on from Olga Pollex's conclusion and move on to the explanations in which various areas of mathematics are discussed in sign languages.

Outlook on the Didactics of Algebra, Geometry and Stochastics in Sign Languages

In this chapter of the handbook, we have taken a look at the theory and practice of teaching mathematics, and in the last section we have seen a number of excellent concrete examples from different areas of mathematics. These examples show impressively how mathematics and Sign Languages can be linked. In the following chapters, research results from didactics of mathematics that take into account the special needs of deaf learners are presented. In Dialog between researchers and practitioners we would look for new ways to link algebra, geometry and stochastics lessons on both a theoretical and practical level. The mathematical tasks or problems developed by Sign Language teachers and researchers in video format can be supplemented and used as tools for diagnosing and promoting vivid-imaginative and verbal-logical thinking with objects, pictures and animations as required.

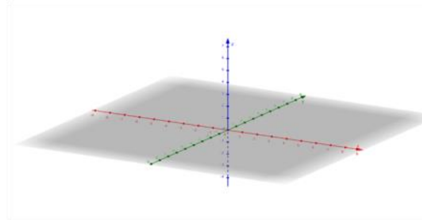
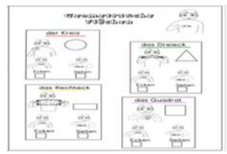
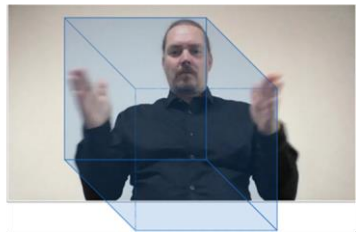
Before we move on to more specific examples of how this can be done in the classroom, we would like to look at more specific empirical findings that provide empirical evidence of the importance of mathematical support in sign languages. We will ask ourselves for example very concrete what does "visual-imaginative" exactly mean in the context of geometry. At this point, it should be briefly mentioned that it is about the ability to use images, sketches and models that can be perceived visually or through other senses, initially as supports and mediators for mathematical thinking and reasoning, and then to gradually detach oneself from them and operate with invisible or sensually imperceptible images mediated by signs or words.

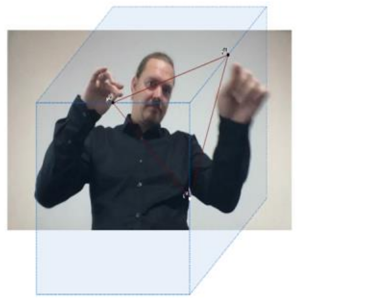
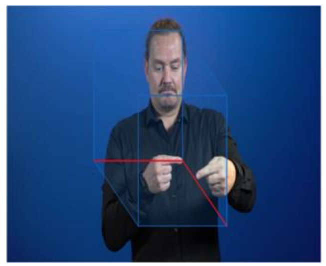
Terminology (Mathematics)

Terminology table can be found in the appendix *Terminology (Mathematics).*

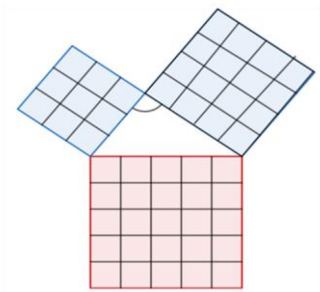
Appendix

Terminology (Mathematics)

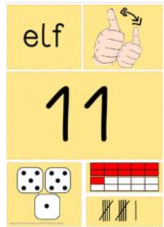
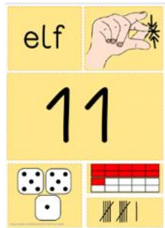
	Terminology	Definition	Examples	Illustration
1	Plane	A plane is a two-dimensional subset of three-dimensional space. A plane is an infinitely large surface which completely contains every line connecting any of its points.	A plane Illustrated as a xy-plane	
2	Plane figures	A figure is a plane if all its points lie in a plane. Plane figures are often bounded by lines. The non-bounded exceptions are, for example, points, rays or angles.	A two-dimensional geometric figure (triangle, square, circle) The boundary surface of a three-dimensional solid. In our example illustration, each square (plane figures) lie in different planes.	 Copy right: Frau.TAUBe  Copy right: Tino Sell, Swetlana Nordheimer

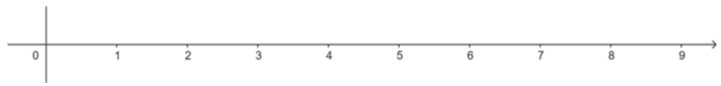
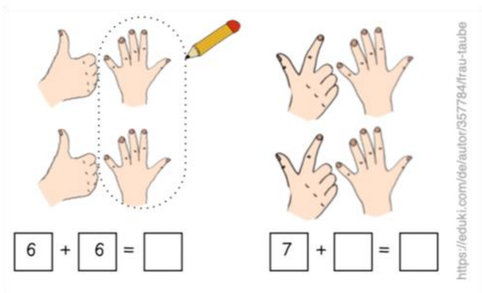
3	Sides and edges in 2D and 3D	Sides of plane figures are straight line segments that delimit plane geometric figures. Edges of three-dimensional solids are straight line segments that delimit boundary surfaces of a three-dimensional solids. Edges can also be described as intersections of two plane figures. For example, squares.	Sides of triangles Edges of cubes	      Copy right: Frau.TAUBe Copy right: Tino Sell, Swetlana Nordheimer

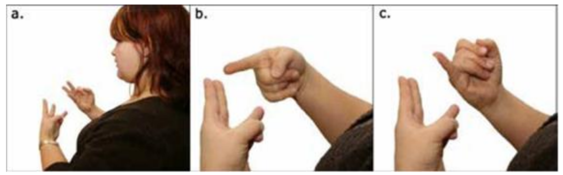
4	Corner and vertices in 2D and in 3D	The corner or vertex is a distinct point of the boundary line or boundary surface of a plane figure or three-dimensional solid. The corners of two-dimensional polygons are the points at which the sides meet. The vertices of three-dimensional solids are points where at least three planes meet.	For example, a triangle has three corners. In our illustration the corners of the triangle are marked with letters: M_1, M_2, and M_3. Three squares meet in every vertex of the cube. The cube has eight vertices.	

5	Pythagorean Theorem	In a right-angled triangle, the area of the square built on the hypotenuse (c) is equal to the sum of the areas of the squares built on the cathets. (a and b): $a^2 + b^2 = c^2$.	For example, in a right-angled triangle with hypotenuse $c = 5$ cm and cathets $a = 3$ cm and $b = 4$ cm is true: $3^2 + 4^2 = 5^2$ $9 + 16 = 25$	 Copy right: Swetlana Nordheimer

6	Base 5	In German Sign Language, and some other sign languages, the numbers 1 to 5 are represented with the dominant hand. The number of fingers represents the corresponding cardinal number. If the number 6 is reached in the number sequence, it is represented as 5 with the non-dominant hand and 1 with the dominant hand. Similarly, the five in German Sign Language must also be represented with the non-dominant hand for the numbers 7, 8, 9 and 10, with both hands representing a 5 for the 10.	Our illustration shows the numbers 6 and 11 in German Sign Language and Ukrainian Sign Language. In UkrainianSign Language both the number 6, and the number 11 refer to the base 5. 11 is represented by 5 + 5 +1. One 5 is represented by the non-dominant hand. Another 5 is represented through four fingers of the dominant hand together, with one finger extended for the 1. The representation of the 10 in Ukrainian Sign Language is partly symbolic, as 4 fingers together symbolically replace number 5.	 Copy right: Maike Beyer

7	Dialectal sign variations as source of confusion	Due to the dialects of national sign languages, different signed numbers can exist and be known to the learners. The differences can cause confusion with mathematical rules.	In the Berlin dialect, the numbers 11 and 12 are mathematically confusing. In this dialect, the number 11 is represented by tapping the thumb and index finger together. 11 is often confused for 12. The number 12 is represented by tapping the thumb, index, and middle fingers together. 12 is often confused for 13. To avoid these confusions, one options to represent 11 and 12 the same as 13 – 19, which is by shaking the number which is added to 10. When using simplified or alternate versions of numbers for clarity, it is also important to show the linguistically correct version. For students with language deprivation syndrome or learning difficulties, alternate versions of signs maybe more effective.	 Simplified version (Copy right: Frau.TAUBe)  Linguistically correct version (Copy right: Frau.TAUBe)

8	Successor	The successor is the next largest natural number.	The successor of 5 is 6.	 The relationship of a successor can be illustrated with number line.
9	Doubling algorithm	Doubling algorithm uses a base hand shape '5' as a number line in German Sign Language to make calculation easier.	For example, to double 6, both numbers are signed in the air. Both fives are bundled together to form a 10 and the remaining numbers are added.	 Copy right: Frau.TAUBe

10	Signed algorithms	Deaf people use fingers, both hands and three-dimensional neutral space systematically to add, subtract, divide, and multiply. Fingers, hands, and their movements in space have special roles where each element is used as a buoy when calculating and anchoring. For example, visual representations of totals and subtotals support mental calculations.	Calculating 3 x 8 in Finnish Sign Language and other examples are described by Rano (2018)	 Calculating 3 x 8 in FinSL (Copyright: Rano, 2018)

11

Didactical terms (mathematics)

Extended model for mathematical development of deaf learners based on Yashkova (1988) and Rozanova (1971/1978, 1991) with a focus on Sign Languages. Developed in cooperation with Olga Pollex, Swetlana Nordheimer and Viktor Werner

1. Visual-active level
   - Understanding abstract concepts through practical actions and mathematical games mediated through sign languages

2. Visual-imaginative level
   - Operating with visual and mental images mediated through signs and words.

3. Verbal-logical level
   - Articulating mathematical proofs on different levels of thinking (visual actions, icons and symbols) mediated through signs and words.

<table>
<tr><td colspan="3">1. Visual-active level
• Understanding abstract concepts through practical actions and mathematical games mediated through sign languages</td><td rowspan="2">Problem situation
Goal: The chicks should have as much free space as possible.</td></tr>
<tr><td>Russisch</td><td>Deutsch</td><td>English</td></tr>
<tr><td>Первая стадия (наглядно-действенное мышление) формируется в процесе практической предметной деятельности. Уже в конце первого года жизни дети способны к эмоциональному переживанию потребности решить несложные практические задачи, которые даны им в наглядной форме. Функция наглядно-действенного мышления заключается в получении сведений о скрытых свойствах обьекта, выявляемых в ходе практических преобразований.</td><td>Die erste Stufe (anschaulich-handlungsorientiertes Denken) bildet sich im Prozess der praktischen Tätigkeit am Objekt.

Bereits am Ende des ersten Lebensjahres können Kinder, die Notwendigkeit erleben, einfache praktische Aufgaben, wenn diese Aufgaben anschaulich formuliert sind.

Die Funktion des visuell-handlungsorientierten Denkens besteht darin, Informationen über die verborgenen Eigenschaften eines Objekts zu erhalten, die im Laufe der praktischen Umformung zutage treten.</td><td>The first stage

(visual-active thinking) is activated through the process of practical object activity. Already at the end of the first year of life, children can experien ce the emotional need to solve simple practical tasks, which are given to them in a visual form.

The function of visual-active thinking is to obtain information about the hidden properties of an object, which are revealed in the course of practical transformations.</td><td>How can you make a fence from a rectangular piece of cardboard of a given length so that the surrounding area is maximized?

Quelle: Цыплята на рынке в мелкой коробке, почему они не выпрыгивают (Курочка | Дзен (dzen.ru))

This problem can be solved practically by children, as well as adults,

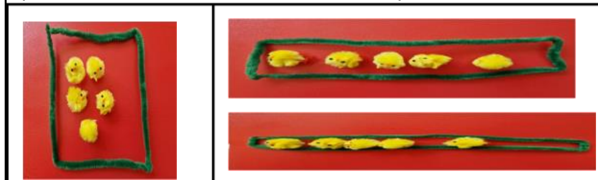
by simply building different fences and estimating the size of the area using practical means.

How many chicks can fit in the space?
How much free space is there?

The area can also be measured using square centimeters, square decimeters or other units.</td></tr>
</table>

Additional remarks: From the beginning, language plays an important role. The questions are formulated and phenomena are discussed in language. These can be video or text recorded and represented in addition to the pictures. In line with Heinz (2000), it can be said that both children's problem solving, and scientific mathematical experiments include “depictive action-oriented thinking”. In this type of thinking, practical problems are described and, in a way, recreated and mediated by language. This applies not only to science, but also to social phenomena, which are often empirically studied and quantified in scientific discussion. Statistical and linguistic tools measure, perform and recreate phenomena. The structure and course of problem-solving activities depends heavily on the language in which research questions are posed. The structure of the visual-action-oriented processes depends on the language in which the leading problem questions are formulated. In this sense, it is also important that sign languages can help determine the structure of problem-solving at schools in general, and particularly in science.

Abstraction in games and hands-on experiments: Even in an identical situation, the formulation of the question of the maximum free space for chicks is structured differently in sign languages than in spoken and written languages. In the classroom, similar real-life problems can be illustrated in stories and playful situations. For example, the children are given little chicks and asked to solve problems in an action-oriented way

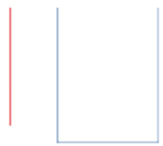


Copy right: Swetlana Nordheimer

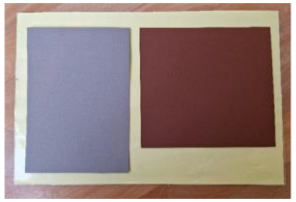
If you choose extreme examples, the choice becomes clear. The area of the rectangle is small if the difference between the sides of the rectangle is large.

Other options for this playful experience could be conducted by drawing the rectangles and laying them out with unit squares. The teacher could also bring a large box into the classroom and have the children play with little chicks inside. This activity could be done outside by marking the boundaries of the base of the imaginary box with chalk or string. This could give the children a new experience of space and area.

Playing requires abstraction. For example, the ability to interpret the rectangles made of pipe cleaners as fences for chicks. Transfers between the areas of “reality of the chicks in the box”, “playing with little chicks and pipe cleaners”, “mathematics”, and other games require signed explanations for sign-language-oriented learners. Here, actions can be directly linked to productive signs that describe them. Even at this level, it is important to use conventionalized lexical signs to enable abstraction processes. Not all signs or words used for play, storytelling, or mathematical conversations can be visualized. The symbolic function of language, especially sign languages, is important for successful mathematical development from an early age.

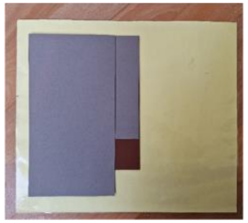
2. Visual-imaginative level • Operating with visual and mental images mediated through signs and words.			**Conjecture:** The square has the largest area among the rectangles with the same perimeter.
Russisch	Deutsch	English	
Язык играет особую роль в переходе к наглядно-образному мышлению. Сначала ребенок решает задачи с помощью практических действий, затем, по мере накопления опыта, вырабатывает более рациональные способы их решения. Тогда решение принимается уже в форме представлений, на основе образов действий. Важной предпосылкой развития наглядно-образного мышления является формиро-вание умения различать реальные объекты, планы и модели, отражающие эти объекты. Посте-пенно дети учатся выполнять мыслительные операции на основе воспринимаемых образов, не манипулируя предметами.	Beim Übergang zum *visuellen-imaginativen Denken* kommt der Sprache eine besondere Rolle zu. Zunächst löst das Kind schwierige Probleme mit Hilfe praktischer Handlungen, dann, mit zunehmender Erfahrung, entwickelt es rationalere Wege zu ihrer Lösung. Dann wird die Entscheidung bereits in Form von Darstellungen, auf der Grundlage von Handlungsbildern getroffen. Eine wichtige Voraussetzung für die Entwicklung des visuellen-imaginativen Denkens ist die Ausbildung der Fähigkeit, zwischen realen Objekten, Plänen und Modellen, die diese Objekte widerspiegeln, zu unterscheiden. Allmählich lernen die Kinder die Denkoperationen angelehnt an die wahrgenommenen Images ohne Manipulationen mit Objekten durchzuführen.	In the transition to visual-imaginative thinking, languages have a special role. At first, the child solves difficult problems with the help of practical actions. Then, as they accumulate experience, the child develops more rational ways to reach solutions. Then, the decision is made in terms of representations, through operating images. An important condition for the emergence of visual and figurative thinking is the development of the ability to distinguish between real objects and models reflecting these objects. Gradually, children learn to perform thought operations based on perceived images without manipulating objects.	Proof: graphic (iconic)  The square outlined in red and the rectangle outlined in blue have the same perimeter. To create the blue rectangle, we shortened one side of the square and lengthened one side of the square.  If we place the two figures on top of each other, they partially overlap. When overlapped, there are also two areas that do not overlap. We call them 'remainder rectangles'. The red remainder rectangle is larger than the blue remainder rectangle. This means that the red square is larger than the initial blue rectangle. The statement above applies to the example, as well as to all rectangles. It is therefore universally valid. The images do not only represent specific cases, but all rectangles and squares with the same perimeter. However, the images are not self-explanatory (Winter) and only acquire their iconic content and mathematical significance through linguistic or symbolic embedding. Accordingly, descriptive visual evidence must be embedded in sign language in order to be meaningful for sign language-oriented learners. Ideally, learners detach themselves from the concrete images and operate with mental images of rectangles (Kruteski, Wittmann, Kadunz).

Proofs with actions (enaktiv)



We cut out a grey paper rectangle and a brown paper square. They have the same perimeter. We want to show that the brown square is larger than the gray rectangle.

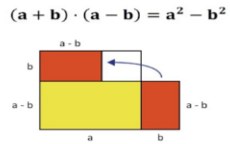
We place the rectangles on top of each other and compare the remaining area. To do this, we cut off the piece of the gray rectangle that does not overlap with the area of the square.

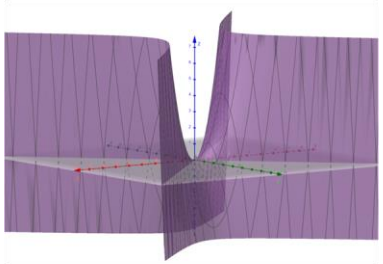
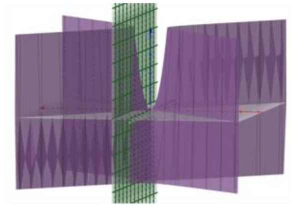
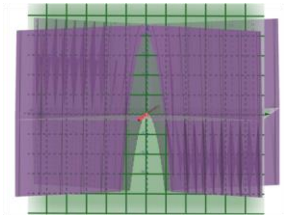
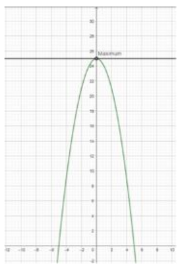
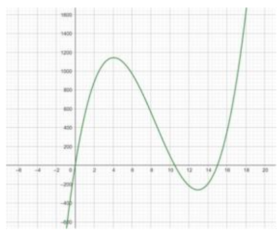


We place the cut-off gray remaining area on the brown remaining area. This creates a difference area in the shape of a square. Therefore, the area of the large brown square is larger than the area of the large gray rectangle.

Didactic remarks: In this activity, the actions with the cardboard rectangles are not self-explanatory. They are only meaningful in conjunction with verbal explanations. Some linguistic explanation can be replaced with video recordings of each step. However, video demonstration must still be accompanied by the question, or the assumption must be formulated linguistically. Furthermore, the linguistic formulation can be supported or supplemented by algebraic symbolism. For example, in her textbook, Kortadze uses the tools of algebra to formulate mathematical statements for 1st and 2nd grade learners without phonetic or written language. Such algebraic or geometric means can, if necessary, mitigate the need for verbal explanations. However, they cannot replace verbal formulations. It is important that the linguistic formulations are perceptible and understandable for the learners. Therefore, the extent of the learner's ability to perceive linguistic explanations visually, auditorily or tactilely, according to their individual perceptual capabilities, must be clarified in advance. Observation by Swetlana Nordheimer: From my experience cooperating with teachers from different schools and researchers from different universities, teachers and researchers are often not aware of how much spoken language they use, even in so-called "proofs without words" (Nelson). In these cases, written versions are not always documented and therefore are not accessible to many learners.

And last but not least: How can iconic proofs be made accessible for deaf-blind learners? Tactile signing and tangible teaching media can be used in place of or to supplement other approaches. In this case, working with enactive media would be particularly relevant. The adaptation of video materials in signed languages may prove challenging. Once the questions about perceptual prerequisites have been clarified, the next step is to identify which languages and modalities are perceivable and understandable by the learners and at what level. With regard to language deprivation, it cannot be taken for granted that all learners and teachers have a command of sign languages at the required level.

3. **Verbal-logical level** • Articulating mathematical proofs on different levels of thinking (visual actions, icons and symbols) mediated through signs and words.			**Formalization through mathematical symbols**
Russisch	Deutsch	Englisch	
Словесно-логическое мышление разделяют на конкретно-понятийное и абстрактно-понятийное мышление. Большую роль в совершении мыслительных операций играют образы, отражающие непосредственный опыт детей. Сначала обобщенности представлений, а потом тренировка в переходах к следующим уровням обобщенности образов, к их усложнению. Наиболее высокая стадия – абстрактно-понятийное мышление, характеризуется способностью человека самостоятельно решать сложные познавательные задачи, обобщенностью, взаимосвязью и обратимостью мыслительных действий, произвольностью в оперировании конкретным и абстрактным материалом, умением контролировать и обосновывать свои рассуждения и выводы.	Das *verbal-logische Denken* wird in konkret-begriffliches und abstrakt-begriffliches Denken unterteilt. Bilder, die die unmittelbare Erfahrung der Kinder widerspiegeln, spielen eine wichtige Rolle bei der Durchführung von Denkoperationen. Zunächst wird die Verallgemeinerung von Darstellungen geübt, dann die Übergänge zu den nächsten Stufen der Verallgemeinerung von Bildern, zur Steigerung der Komplexität. Die höchste Stufe - das abstrakt-konzeptionelle Denken - ist gekennzeichnet durch die Fähigkeit einer Person, komplexe kognitive Aufgaben oder Probleme selbständig zu lösen, durch Verallgemeinerung, Verknüpfung und Umkehrbarkeit von Denkhandlungen, Willkür im Umgang mit konkretem und abstraktem Material, die Fähigkeit, ihre Überlegungen und Schlussfolgerungen zu kontrollieren und zu begründen.	*Verbal- logical thinking* is divided into concrete-conceptual and abstract-conceptual thinking. Images reflecting children's direct experience play a major role in the performance of thought operations. The first step is the generalization of representations. Then there is training to transition to the next levels of generalization of images, with increasing complexity. The highest stage, abstract conceptual thinking, is characterized by the ability of a person to do the following tasks independently: solve complex cognitive tasks, generalize, interrelate and reverse thought actions, understand arbitrariness in the operation of concrete and abstract material, and have the ability to control and justify their reasoning and conclusions.	 Source: [(168) Binomische Formeln - 3. Binomische Formel - Übung 5 | *NEUES KONZEPT* - YouTube] $U_{Square} = a + a + a + a = 4a$ $U_{Rectangle} = 2(a-b) + 2(a+b) = 4a$ $U_{Square} = U_{Rectangle}$ $A_{Difference\ square} = b^2$ $A_{Square} = a^2$ $A_{Rectangle} = (a-b)(a+b) = a^2 - b^2$ $A_{Square} = a^2 \geq a^2 - b^2 = A_{Rectangle}$ Following on from the practical, visual, and linguistically embedded experiments and proofs, algebraic means and language can be used to describe proofs at the verbal-logical level. This can be accomplished to a greater extent than before, supplementing and perhaps replacing other linguistic means. The transitions between geometry, algebra and other areas of mathematics are often mediated by language. In these cases, sign language supplements, explanations and formulations are indispensable for sign language-oriented learners.

$A(x,y) = (x-y)(x+y)$  The function $A(x,y) = (x-y)(x+y)$ describes the change in the area of a rectangle as a function of the side length of the square x and its lengthening of one side of the square by y or shortening of the other side by y.	Assume that the perimeter or side of the square is known. For example, let $x = 5$.  All solutions for $x = 5$ can be represented using the green plane. They lie in the intersection curve between the violet and green planes and form a parabola.	The parabola has its maximum in the vertex, which lies exactly above the x-axis, where $y = 0$.  With regard to sign language embeddings, the representations of functions in 3D are interesting and provide new views. However, one could also start from a concrete scale and examine functions of a variable and their graphs in the plane.
If the perimeter of the rectangle is given and the change in a side's length is described by x, then the value for the maximum area is at the **vertex of the parabola.** 	Even if the maximum of quadratic functions can be determined using the vertex, the means of differential calculus can also be applied here. For example, you could graphically illustrate that the slope of the derivative at the vertex is zero. If, for example, you try to optimize not the area but the volume of a box or carton in which the chicks are sitting, you quickly arrive at the third degree functions, where differential calculus can be helpful.	

Concluding remarks:

At any level of the learners' thinking development, embedding content through sign language is impactful on their mathematical development. The levels are not strictly hierarchical. The "ascent" from the practical-action level to the verbal-logical or symbolic level of thinking is not the only important factor. Especially in everyday life, the problems must be translated back into the non-mathematical world to apply them to non-mathematical reality. For example, the construction of the optimal open-top box for the chicks can provide practical guidance to the conclusion of the original question. The ability to transfer verbal logic to symbolic knowledge or understand concepts constructively is foundational to engineering and architecture. The bridging of different levels of mathematical thinking and areas of mathematics (i.e. geometry, algebra, stochastics etc.) to technical sign languages and didactical aspects is in need of further development.

3. DeafDidaktik-critical View of mathematical Text Tasks

Staudt, B., Sieprath, H., Karar, E., Baklaci, M., Schmidt, D. & Grote, K. (2024) *

Competence Centre for Sign Language and Gesture (RWTH Aachen)

In the context of an empirical study on DeafDidaktik by Staudt (2024) in mathematics lessons with deaf students whose first language (L1) is German Sign Language (GSL), it was repeatedly observed that working on mathematical text tasks is associated with specific comprehension difficulties.

These issues were discussed and analyzed with the DeafDidaktik-Team at the SignGes Competence Centre for Sign Language and Gesture at the RWTH Aachen University under the direction of Dr. Klaudia Grote. Based on these considerations, a DeafDidaktik adaptation of a text task designed for hearing children was developed, resulting in a mathematical text task tailored to the needs of deaf children. To assess the efficacy of the adapted task, a preliminary empirical study was conducted, in which a 'kangaroo task' was presented to two ten-year-old children: one hearing and one deaf.

Note: The so-called 'kangaroo tasks' are an example of a pedagogical approach that originated in the Australian education system and has since been adopted in a European mathematics competition. The kangaroo tasks have been employed in Australian educational institutions since 1978, with their implementation in German schools following 4 years later. The objective of these tasks is to provide support and challenge for students in the third and fourth grades regarding mathematical learning (for further information, see https://www.mathekaenguru.de/international/index.html—09.11.2024).

The text task from the 2021 Kangaroo Competition, which Staudt introduced to the schoolchildren in a preliminary study, is as follows: **In a modest cinema, five companions occupy an entire row. Paul is not seated in the fifth position. Anabel, on the other hand, has selected the first seat. Lynn is situated between Joshua and Selin. Thus, the question arises as to the precise location of Lynn's seating.**

The hearing child with German as their first language (L1) solved the task promptly and accurately. The deaf child, whose first language is German Sign Language (DGS), acquired at a relatively late stage, and German, which may be considered a second or even third language acquisition due to the Russian migration background of the parents, experienced considerable difficulties in reading and understanding the above text task. Subsequently, the child was presented with a translation of the task in DGS.

The child's feedback and reactions suggest an enhanced comprehension of the signed task. Nevertheless, despite the translation into DGS, the child could still not comprehend the mathematical methodology for completing the task.
Subsequently, the child was presented with an animation of the content, designed following the principles of DeafDidaktik and edited with the software PowerPoint. Immediately following the initial presentation, the child demonstrated an understanding of the context depicted. In a second iteration, a brief signed explanation of the task was additionally provided, facilitating comprehension of the mathematical approach and enabling the child to complete the task.

For the DeafDidaktik-version of the text task, DeafDidaktik-principles had to be applied, which required a three-phase DeafDidaktik-analysisin advance. The final presentation of the material included videos in German Sign Language (DGS) and PowerPoint slides with corresponding animations and transitions, each incorporating Principles. These included an inductive style of explanation, subject-object buoys, a signed elimination strategy, localization, and changes in perspective. This was achieved by applying sign classifiers and, in addition, constructed action (CA) or constructed dialogue (CD) (Grote, Sieprath, Staudt, Fenkart & Karar – Work in Progress 2024). Furthermore, elements of DeafScience were incorporated, including the presentation of sign language videos in circular formats with color-coded frames to differentiate between them. In this case, the color 'white' represents the introduction of the task, 'blue' represents additional explanations, 'red' represents the question, and “green” represents the answer or solution (Sieprath et al., 2024).
This preliminary study indicates that deaf students encounter various challenges when solving mathematical tasks in a written form. These text tasksrequire the students to employ a variety of decoding procedures or processes, including decoding the content, translating the written text into mathematical codes, and solving the mathematical problem.
Considering the findings of this preliminary study, this video presents the initial criteria for creating signed DeafDidaktik videos for mathematical tasks. However, it is essential to note that these criteria require further empirical investigation in educational contexts.

4. Signing about Variables and Equations

Angeloni, F. & Hausch, C.

Introduction

Bilingual practice with a sign language and a written language is fundamental in teaching sign language-oriented pupils. However, the characteristics of sign languages should also be considered in education research (Grote et al., 2018). Over the time studies have shown that sign languages can influence the teaching and learning of mathematics in such a way that significant differences to spoken language practice can sometimes arise. Some recent examples of such studies in mathematics education can be found in Krause (2017) and Wille (2020). It has also already been shown, for example, that "[...] the use of sign language space in the mathematics class can have a decisive function, e.g. [...] in the acquisition of specialist and technical sign language signs that do not (only) consist of certain signs for specialist and technical terms of [a] spoken language" (translation from Angeloni, 2023, p. 532).

In this chapter, basic notions and concepts of elementary algebra – such as "variable", "equation", etc. – are considered from a sign language perspective based on results of a broader project on teaching and learning elementary algebra in a sign language (cf. Angeloni et al., 2022; 2023). In the first section, the investigated variable aspects and a central property of sign languages, iconicity, in mathematics are presented. Then, key principles for teaching in a sign language and the learning environments that were used in the project are presented. Selected results on the object aspect, the substitution aspect and the shell aspect of variables are explained and possible implications for mathematics teaching are discussed. Unless otherwise stated, the signs presented here are signs of Austrian Sign Language (ÖGS).

Variable Aspects

Variables are various and can be viewed from different points of view: *Object aspect*, *substitution aspect*, *calculus aspect* (Malle, 1993) and *shell aspect* (Wille, 2008). Under the object aspect, a variable is defined as an unknown or unspecified number (Malle, 1993, p. 46). According to the substitution aspect, a variable functions as a placeholder in which numbers may be inserted (Malle, 1993, p. 46). The placeholder vanishes when a number is assigned to it. This placeholder remains under the shell aspect that means a variable is like "a cover or a box for the number but it is still here" (Wille, 2008, pp. 422-423). According to the calculus aspect, a variable can also be only a sign "with which one may operate according to certain rules" (translation from Malle, 1993, p. 46). Here we examine how these different variable aspects could be related to each other.

Iconic Sign Language Signs in Mathematics

Fig. 1a *Fig. 1b* *Fig. 1c* *Fig. 1d* *Fig. 1e*

Figure 1. Iconic signs of ÖGS in mathematics

Iconicity is a property common to all sign languages. Signs that show a direct or indirect similarity to the referred are iconic signs. In the first case, the signs are defined as *pictorial*, in the second case as *schematic icons* (Kutscher, 2010). According to Kutscher (2010), pictorially iconic signs are divided into those whose hand shape resembles the shape of the referred and those in which the movement path of the sign resembles the shape of the referred: One example is the sign in Figure 1a, in which the shape of the hands resembles the round brackets around a term. Another example is the sign in Figure 1b, in which the movement path of the index finger of the right hand imitates the shape of a round bracket. Wille, who theorizes as Kutscher iconicity according to Peirce, distinguishes further between “mathematical signs in which the movement path imitates an action on mathematical inscriptions” (translation from Wille, 2020, p. 206) and give as an example the sign ROUND-OFF in Figure 1c (drawn from Wille, 2020, p. 207).

The indirect similarity to the referenced is not realized by the schematic icons through the hand shape or the movement path, but via schemata: “knowledge structures [...] that make it possible to interpret experiential data according to cognitively anchored standard patterns of objects, events, situations or action sequences” (translation from Kutscher, 2010, p. 96). These signs are also differentiated according to the type of indirect similarity: 1) Signs with a “metonymic relationship” imitate something that stands for the referred: this is often a relationship from the part to the whole or vice versa. An example is the sign MATHEMATICS (Fig. 1d), which – with the exception of regional variants – differs from the sign NUMBER only in the mouth image. The metonymic relationship exists to the extent that “in mathematics (among other things) one deals with numbers” (Wille, 2020, p. 202). 2) Signs that imitate an action, typically manipulating what is being referred (Kutscher, 2010): The sign CALCULATOR in Figure 1e (drawn from Wille, 2020, p. 203) is an example for that. Here, the action of typing is imitated. This means that both the calculator and the typing itself can be expressed. 3) Signs that imitate an action from which the referred results: The sign NUMBER (Fig. 1d) again is an example of that: if a number can be seen as the result of counting, so the movement of the sign NUMBER imitates “counting in sign language”

(Wille, 2020, p. 202). Iconicity can also be intra-linguistic: signs exhibit similarities to other signs. In the context of calculating interest, an example in DGS (German sign language) is the sign for “credit”, which is like the sign for “to lend” (Krause, 2016). This example can also be found in some regional variants of ÖGS.

Teaching in a Sign Language

For the bilingual mathematics class with a sign language, “various modality-related structural differences between spoken and signed languages should be taken into account” (translation from Grote et al., 2018, p. 435). This includes a higher degree of iconicity in sign languages than in spoken languages, due to which there is a stronger coherence between the signs and the properties of the referred, which “can be experienced directly in a sensory – embodied – way” (translation from ibid., p. 428). In mathematics classes, therefore, “the type of explanations should correspond with the iconic aspects [...]” (translation from ibid., p. 433). Another characteristic is *centering*: “A [...] topic is [...] placed at the center and [...] a syntagmatic context is established” (translation from ibid., p. 429). This means that different concepts that are used together are placed in relation to each other around a central concept. According to Krause (2016, p. 578), “car”, “road”, “drive” and “fast” are examples of concepts that can stand in a syntagmatic relationship. A syntagmatic context can be created by changing the perspective, i.e. switching from one specific concept to another in order to describe it “in more detail” (Grote et al., 2018). This would mean in the mathematics class, for example, that a specific central topic should be placed at the center around which further “knowledge units that can be experienced with the senses are placed” (ibid., p. 430). This raises the question of how coherence would manifest itself in signing about variables and what should be placed at the center.

The Design of the Study

In a 60-minute session, the participants, adults with ÖGS as their basic language, work in groups of three to four on a learning environment with various tasks according to the “think-pair-share” principle (cf. Ruf & Gallin, 1999). The study comprises several learning environments, each focusing on one aspect of variables, and each session comprises only one learning environment. The tasks are designed in ÖGS and videos are used instead of a task sheet. The sessions are also accompanied in ÖGS and recorded on video. The video material is evaluated in two phases: In the first phase, relevant passages are documented with glosses, images, possible translations (to German) and video excerpts. Interviews are conducted with the participants based on this documentation. The whole process from developing the learning environment to the evaluation of the video material takes place in ÖGS.

The Learning Environments

Fig. 2a

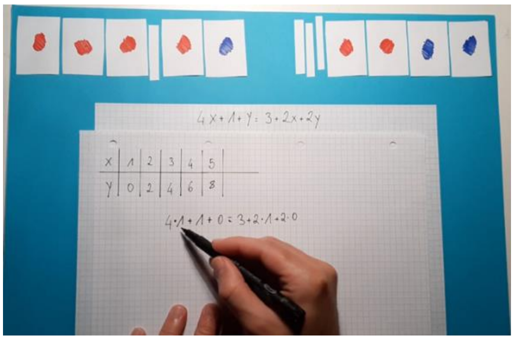


Fig. 2b

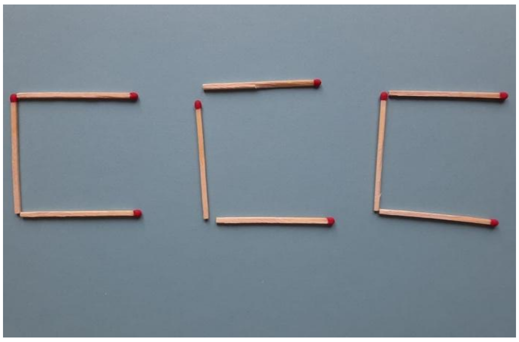


Fig. 2c

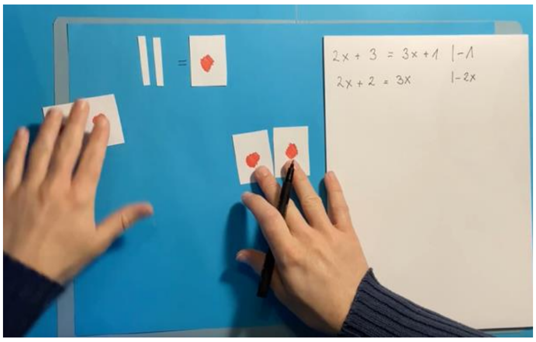


Fig. 2d

Figure 2: Excerpts from the videos that are used as tasks

The tasks are based on the „Knack-die-Box“ (en. *crack the box*) learning environment (Affolter et al., 2011): Blue and red boxes contain an unknown number of matches, but two boxes of the same color always contain the same number of matches. Equations as in Figure 2b are formed from such boxes and single matches. The following applies: The total number of matches in the left-hand arrangement should be the same as in the right-hand arrangement.

Two learning environments focus on the object aspect (Angeloni, 2023): One learning environment comprises three matches that are placed one after the other in a c-shape (Fig. 2d). After three matches are placed, the video asks about the total number of matches. After the fifth time, only the fact that some matches (in the c-shape) have been placed again is signed, but not how many. The question about the total number is asked again. In the second learning environment, an arrangement of boxes and matches is signed in each of the tasks (Fig. 2a) and the participants are asked how many matches are in a red box and how many in a blue box.

In the learning environment focusing on the substitution aspect, each time an arrangement of boxes and matches is shown in the video together with the corresponding equation and a table with values (Fig. 2b) (Angeloni, submitted). The video shows that one value for x and one for y from the table are substituted into the equation and the resulting expression is simplified as far as possible. The tasks in the learning environment about the calculus aspect are similarly designed: an arrangement of red boxes and matches was shown in the video together with the corresponding equation (Fig. 2c). Then an equivalence transformation was always carried out first on the equation and then on the arrangements of boxes and matches. Once the solution has been obtained, the students are asked about the relationship between the equation and the arrangements of boxes and the transformations.

A possible Relationship between Variable Aspects

Although the learning environment on the calculus aspect was implemented, the evaluation of the data collected has not yet been completed. A selection of results from the other learning environments is therefore presented below. The signs are ÖGS signs that the deaf participants signed in the respective surveys, as well as findings on the use of these signs from the interviews with the participants. The signs are labeled with glosses. Figures 3a to 3d, 3f and 3g from Angeloni (2023, p. 531) and Figures 4a to 4e from Angeloni (submitted) are also taken up and explained further.

The Object Aspect

Fig. 3a *Fig. 3b* *Fig. 3c* *Fig. 3d* *Fig. 3e* *Fig. 3f* *Fig. 3g*

Figure 3. Signs in relation to the object aspect of variables

The signs in Figure 3 express different facets of the object aspect of variables: The sign EMPTY (Fig. 3a) was used to express that the variable does “content” any number. The number is therefore not present. The sign in Fig. 3b conveys the information that the number has not yet been assigned to the variable or has not yet been communicated. The sign OPEN in Figure 3c expresses that it is unknown whether and which number a variable will assume.

The signs in Figures 3d to 3g are directly similar to the sequence of c-shaped matches, so that these signs are pictorially iconic: The c-shape of the sign in Figure 3d resembles the shape in which three matches are placed each time. On the one hand, the movement of the sign resembles the shape of the sequence in which the c-shapes of matches are placed (cf. Fig. 2d) and, on the other hand, the action of placing twice three c-shaped matches. The sign in Figure 3e can be similar explained, but the hand shape only emphasizes a certain characteristic of the c-shape, namely that the c-shapes can be understood as two-dimensional figures. The sign in Figure 3f describes the number of matches that make up a c-shape in the sequence, and its movement also resembles the shape of the sequence in which the c-shapes lie. Therefore, a pictorial iconicity can be observed here. The signs can also convey further information: The sign in Figure 3d also expresses the exact number (two) of c-shapes and in Figure 3f it is signed that four c-shapes are lying or being laid. The fact that there are exactly four can be deduced from the context in which it was previously signed that three c-shapes are already in the sequence and another one is being added. Without this context, the exact number would not be apparent, as a triple or more frequent repetition of a sign generally only expresses a plural. The exact number would usually be signed beforehand. The signs in Figure 3e and 3g express that the sequence continues indefinitely. In the sign THREE (Fig. 3g), the movement also stretches the range for the number of c-shapes in the sequence and the hand shape conveys that the total number is a multiple of three.

The Substitution and the Shell Aspect of Variables

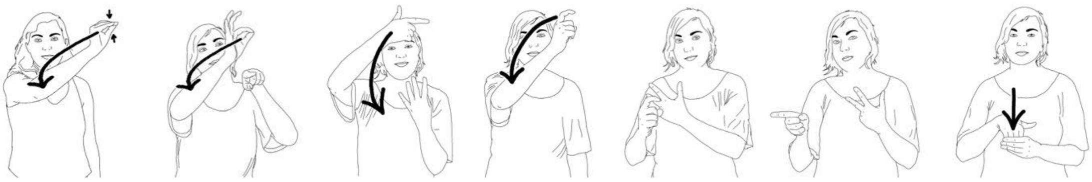


Fig. 4a *Fig. 4b* *Fig. 4c* *Fig. 4d* *Fig. 4e* *Fig. 4f* *Fig.. 4g*

Figure 4: Signs in relation to the substitution aspect of variables

The signs in Figures 4a to 4d are pictorially iconic because they are directly similar to the written image with the table of values above the equation (Fig. 2b). In addition, the first two signs (Fig. 4a and 4b) imitate an action in which something is taken from the table and is placed where the equation is. This means that a number from the table is substituted into the equation. This action can also be imitated with the corresponding number sign, as in Figure 4c, in which it is signed that 2 is substituted into the monomial $4x$. A similar action can be seen in the sign in Figure 4d, but the hand shape is that of the “x” from the finger alphabet. This creates a metonymic relationship between x and the value from the table that is to be substituted into the equation. These signs can therefore be classified as schematic icons.

The sign construct in Figure 4e consists of the number sign THREE behind the sign X. This expresses the shell aspect of variables: The sign X acts as a placeholder into which the number THREE is inserted and remains. The shell aspect is also “visible” in the sign CONTENT (Fig. 4g), which metaphorically expresses that a variable is like a container that can contain a number and can be counted to the schematic icons. This can also be seen in the sign EMPTY (Fig. 3a), which expresses that a variable “does not contain a number”. The sign in Figure 4f is a pointing sign and is used in different ways:

Sometimes it was used to point to the location of the term in the sign language space where three is going to be substituted for x, and sometimes to point to the individual locations in the sign language space where x was previously placed. Neither a direct similarity to the picture with the equation nor a schematic iconicity could be determined here.

Discussion about the Centrality of the Shell Aspect of Variables

The results presented here show that there are different signs that can be used to sign about variables and actions with them. For a variable under the object aspect, i.e. when a variable stands for an unknown or unspecified number, three signs (Fig. 3a to 3c) were observed that express different facets of this aspect. As already explained in Angeloni et al. (2023, p. 4223), the NOT-YET sign conveys that the number for which the variable stands is unknown. Angeloni (2023, p. 530) adds that the number will “become known” and that the sign EMPTY (Fig. 3a) expresses that the number remains unknown. However, there is another difference between the two signs NOT-YET (Fig. 3b) and EMPTY (Fig. 3a), which can be derived from the use of these signs outside the mathematical context. The sign EMPTY indicates that something is empty. If the sign CONTENT (Fig. 4g) – according to that a variable would be regarded as a container – is considered in relation to the sign EMPTY, then the unknown nature of the number would be expressed in the form of an empty container (an empty variable), which places the shell aspect in the foreground. This aspect can also be observed in the case of the substitution aspect, for example by the sign construct in Figure 4e. The extent to which the shell aspect will remain in the foreground can be further investigated in the learning environment about the calculus aspect. The possible influence of the learning environment “Knack-die-Box” should be considered here, because the consideration of a variable as a box or as a shell is already suggested there and sign languages have a wide range of possibilities to refer to something, not least because of iconic properties.

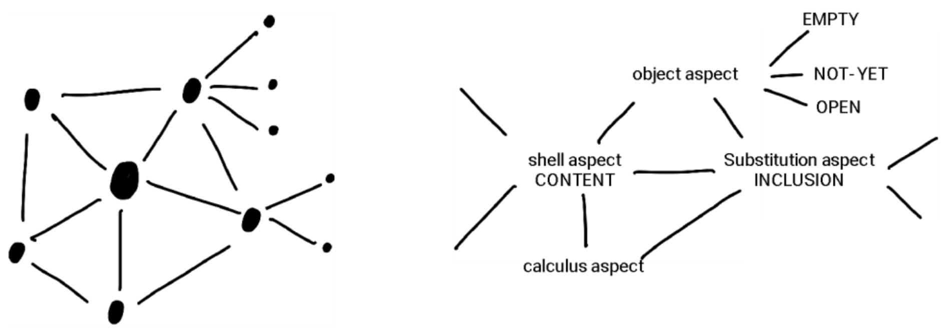


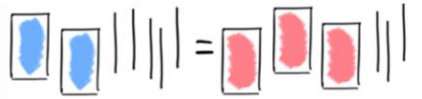
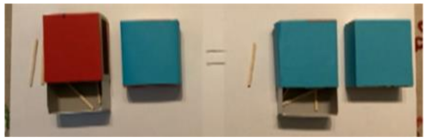
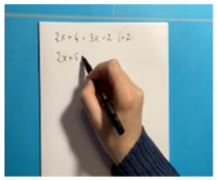
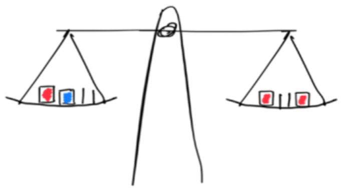
Figure 5: Network of the aspects of variables

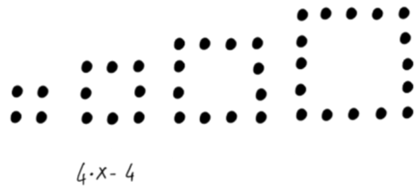
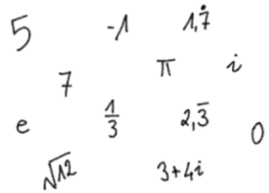
Assuming that the shell aspect plays a key role, this could mean for the practice of mathematics teaching that the shell aspect could be seen as a “central” variable aspect around which “further variable aspects” can be located and thus all aspects can be placed in a syntagmatic relationship to each other. From this perspective, for example, the object aspect, the substitution aspect and the calculus aspect would be regarded as the “further variable aspects”. Furthermore, there would be different facets for the individual aspects, which can be expressed with signs of different iconic types (e.g. Fig. 4b and 4e as well as 3e and 3g) and have a certain direct sensory (embodied) experienceability. This can be observed in the fact that some of the signs from the learning environment about the substitution aspect have a (schematic) similarity to the action of substituting in the foreground. A resulting network could look like Figure 5. In this network, it would be possible to switch the perspective from the central shell aspect to another aspect or a specific facet and take a “closer” look at it.

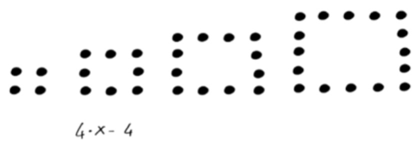
Figure 6: a possible sign (of ÖGS) for “substitution”

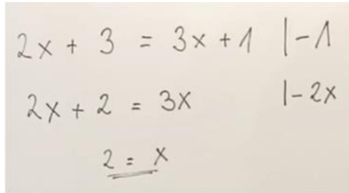
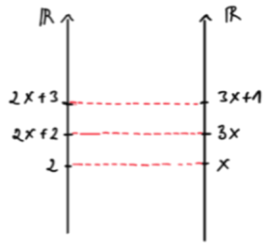
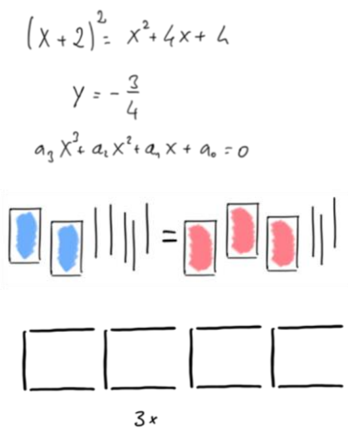
In mathematical sign language discourse, the shell aspect can also serve as a buoy that can be repeatedly referred in order to express further properties of the variable. In addition, the centrality of the shell aspect suggests that the signs INCLUSION (Fig. 6), which has a high iconic coherence with the sign CONTENT, could also be used as a “general” technical sign for the notion “substitution”. The iconicity, the possibility of centering an aspect as well as the possibility of changing the perspective and the consideration of further characteristics of sign languages could also suggest that such an organization of the variable aspects as presented here in other sign languages is possible. On the lexical level, where there are differences in the various sign languages, at least the signs CONTENT and INCLUSION can be found in a similar form in some other sign languages (cf. Sign Language Dictionary, 2018).

Algebraic Terms

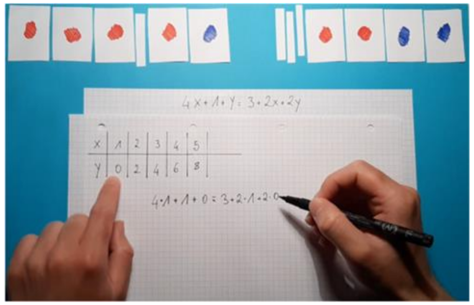
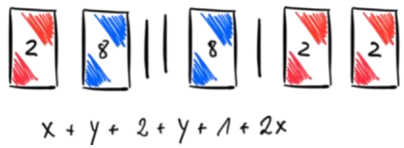
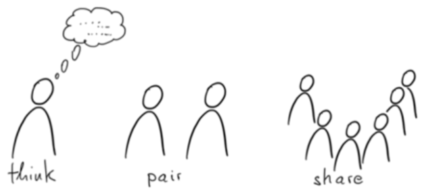
Terminology	Short definition	Example	Illustration
variables	Variables are tools for the general representation of matters.	Variables are usually written with single letters such as x, y, z etc. and can have many different meanings. They can stand for an unknow number, or they are only a symbol to calculate with according to specific rules.	Variables can occur in different mathematical expression such as $x + 3y = 4 + y$ or in as a sequence of boxes and matches  
equations	An equation consists of a term, an equal-sign and a term.	There are many different types of equations. An Example for an equation looks like the following linear equation $2x - 6 = 12$	 

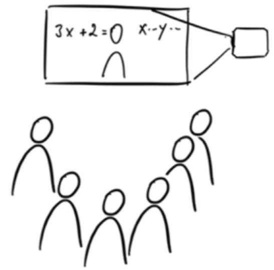
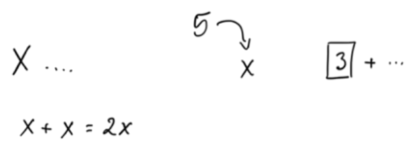
terms	Constants and variables are terms. The result of an operation with terms is also a term.	For example, a term can consist of numbers, a variable and a basic mathematical symbol such as in the following term $2x-6$	$2x-6$ $\frac{2}{x}$ x^2-6 
value	Number	-10	 Terms can have also a value. So if you replace x with 1 in the term them you obtain the value.
substitution	Substitution consists of replacing a variable by a number.	We assign the number 4 to the variable x in $2x-6$ We get 2.	We assign the number 2 to the variable x and the number 1 to the variable y in the term 3x – 5 + 2y. We get 3*2-5+2*1. Now, we can calculate the result. This is 3.

Sequence	A list of consequently numbered objects.	There are a lot of different sequences such as the sequence of positive odd integers 1, 3, 5, 7, 9, 11, 13,... and the sequence of positive integers divisible by 3 from 1 to 20 3, 6, 9, 12, 15, 18	Sequence of numbers 1, 2, 3, 4, 5, 6, 7... 2, 4, 6, 8, 10, 12 ...-20, -15, -10, -5, 0, 5, 10, 15, 20  Sequence of numbered circles 

calculus	A set of rules to solve specific problems.	Solving an expression with different operations and terms such as $2x - 6 + 3x + 10$ according to specific rules. Another problem can be solving an equation. For example, there are different rules for the equation $2x + 3 = 3x + 1$ that can be applicate to calculate the value of the variable x so that if you substitute this value for x and solve the operations on both sides of the equations the same result must be given. In this equation, we can subtract the number 1 on both sides of the equation. The equation becomes the one in the second row. We can here subtract the term 2x on both sides of the new equation and obtain the solution for the equation in the first row.	
elementary algebra	Elementary algebra deals with variables, terms, equations, inequations and systems of equations or inequations.	The chapter linear equations in the secondary school.	

Didactical Terms

Terminology	Short definition	Example	Illustration
Object aspect	A variable stands for an unknown number.	The variable x stands for the number of matches in a red box, but we do not know how many matches are in the box.	The variable x stands for the number of brochures that a person is going to deliver.
Substitution aspect	A variable is a placeholder for a number that vanishes when a number is assigned to it.	We assign the value 6 to the variable z.	
Shell aspect	A variable is like a cover for the number that is still here.	The sign construct in the Austrian sign language with the sign THREE behind the sign X.	
Think-pair-share	Method for teaching that improves the communication between the students.	In the mathematic class the students get a task. Each student tries to solve it and explains her/his own possible solution to another student. At the end the whole class deals with the discussed solutions from the pair discussions.	

Learning environments	A set of tasks with a specific learning target.	The students get a first task with red and blue boxes and some matchsticks that build an equation. The students must solve this equation by moving sticks and boxes. In the second task they must write the equation with symbols that matches the equation with the boxes. In a third task the students must invent some other box-equations and swap this with each other. More tasks are possible.	
Iconicity	Iconicity can be seen as the similarity of a sign language sign to the referenced object.	The sign of the Austrian sign language for a round bracket.	 The sign and the symbol for an open round bracket
Variable aspects	Variable aspects describe what you can do with variables.	For example, a variable can be written with single letters, but also with words or other symbols. A variable can stand for a number that we do not know, we can set a variable equal to a number and we can substitute a variable with a number. We can also see a variable as a symbol with which we calculate according to specific rules.	

5. An Example of a Task from Ttochastics with German Sign Language (DGS)

Warmuth, E., Nordheimer, S. & Sell, T.

Introduction

This article presents a task for primary school Math's lessons that lies at the interface between geometry and stochastics and is primarily intended to contribute to stochastic thinking. Geometric knowledge and skills are a prerequisite. The first author of this article is familiar with the didactics of stochastics and was unable to find any articles on the problems experienced by deaf learners in connection with stochastics during a (brief) search on the Internet. This article should therefore be seen as a suggestion from a layperson and feedback is welcome.

The Importance of Stochastic Thinking

Random phenomena are an integral part of our world. Tomorrow's weather cannot be predicted with certainty, the lottery numbers defy prediction, agricultural yields fluctuate, whether we catch the flu or not depends on many coincidences. Opinion research institutes do make predictions about the outcome of the next election, but they are by no means certain and have often been very wrong. We are confronted with statements about random phenomena almost every day and very often we have to make decisions under uncertainty. As responsible citizens, we need stochastic thinking to be able to interpret such statements sensibly and make well-founded decisions. The term 'Stochastics' comes from Latin and means *the art of skillful conjecture*. The mathematical discipline that deals with this is Stochastics. Due to the importance of basic stochastic education, elements of stochastics are firmly anchored in the educational standards from the beginning to high school graduation. However, this continuous line from first grade was only completed with the *conference of ministers of education of the German stats or* Kultusministerkonferenz (KMK, 2004) resolution on educational standards at primary level.

Jäger and Schupp (1983, p. 15) justify the inclusion of elements of stochastics in primary school lessons as follows: "Similar to the development of the concept of numbers, the understanding of stochastic phenomena, combined with a concept of probability, is formed in a long-term, phased process. The development of stochastic thinking largely takes place during the period in which pupils attend primary and lower secondary school." The upgrading of stochastics in education policy means that it is now anchored in curricula and textbooks nationwide, from primary school through to

A-levels. In our opinion, the 2012/2014 edition of the *Zahlenbuch* which can be translated as *number book* is an outstanding example of the implementation of educational standards at primary school level. We therefore refer you to comprehensive information on the basic concept and materials (including a free download of *Zahlenbuch* 1 to 4) at www.mathe2000.de.

Experience with Combinatorics Tasks in Sign Language

In December 2022, a collection of tasks in German Sign Language was created in cooperation with the "Kangaroo of Maths" competition team and a group of deaf teachers, students and STEM researchers. They are published at https://www.mathe-kaenguru.de/advent/gebaerden/index.html. The deaf teachers and students were able to independently select sub-areas of Maths for the tasks. It is interesting to note that combinatorics was one of the most popular subject areas. The deaf students and teachers enjoyed formulating combinatorial tasks in German sign language. After the teachers had tried out all the tasks in class, one teacher told us: "The Advent calendar made me realize that I haven't addressed combinatorics enough in my lessons."

We would like to build on this observation and present a context that not only contains interesting combinatorial questions, but also offers opportunities to introduce basic ideas of probability theory. By choosing a geometric context, we want to address the (supposed) strengths of deaf learners known from specialized literature and to facilitate their access. Hänel-Faulhaber et al. (2023) state: "Relative strengths can be observed in hearing-impaired children and adolescents in the area of geometry (Pagliaro & Kritzer, 2013; Edwards et al., 2013), which is often associated with the children's strengths in visual perception (Marschark & Knoors, 2012)." However, it should be critically noted that the first two studies have a very small sample size, and the third study was conducted in a university setting. Caution is therefore required when drawing conclusions. In addition, the current collection https://stemsil.eu/mathe-adventskalender/?lang=de also contains many tasks relating to the geometric topic of "cubes".

The Potential of the Beetle Task

The task presented in this article is a modification of the 4th task of the 5th example in the educational standards at primary level (KMK 2004, p. 20). There, the task consists of finding all the shortest routes from A to Z. It is mainly assigned to the key idea of *space and form* and is located in requirement area III. The reference to the standards is established as follows:

- apply mathematical knowledge, skills and abilities when working on problem-based tasks
- recognize, use and transfer relationships to similar situations, recognize, describe and use spatial relationships (arrangements, paths, plans, views)

We see the additional potential of this task firstly in a combination of geometric and stochastic ways of thinking. We thus provide an example of meaningful networking. This is now a recognized principle in mathematics lessons. Roth (2013, p. 1) writes: "For mathematics as the science of patterns (cf. Devlin, 1998, pp. 3-4) and structures, it is characteristic that it searches for relationships between phenomena or consciously establishes them. [...] In addition, it is essential for learning success to recognize or establish relationships between the various content areas and to apply acquired knowledge and skills. In addition, it is advantageous in building understanding if relationships between phenomena, representations, terms, concepts, contexts, etc. are established or at least specifically sought." In addition to networking, the task offers an excellent opportunity for natural differentiation (Wittmann, 2001).

First Modification Beetle Task

A beetle crawls along the edges of a cube. It starts at corner A and wants to reach the opposite corner Z. Because it is so small, it cannot see its destination and at each corner - including A - it crawls in one of the three directions along the entire edge to the end. But he never goes back. After three edges, he is tired and stops.
What do you think, is it more likely that he arrived in Z after three edges or that he didn't arrive in Z? Or are the chances the same?

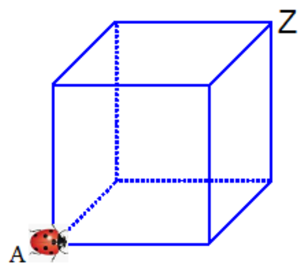


Figure 1: KMK sample task

This is how the problem was presented (slightly modified) by primary school teacher B. Winkenbach (2011) in her field reports on a teaching experiment in her 4th grade class. The terms "Kante" (edge), "Ecke" (vertex), "auf gut Glück" (on the off chance), "wahrscheinlich" (probably) are essential for understanding the task and must be familiar to the children from previous lessons and, if necessary, repeated using suitable tasks.

Further Modification of the Beetle Task

In cooperation with Tino Sell, we have didactically and linguistically revised the task text. The revised task is offered bi-modally in German Sign Language (DGS) and in German. The illustration shows an excerpt from the German Sign Language version of the task, in which the edge model is linked to the directional signs.

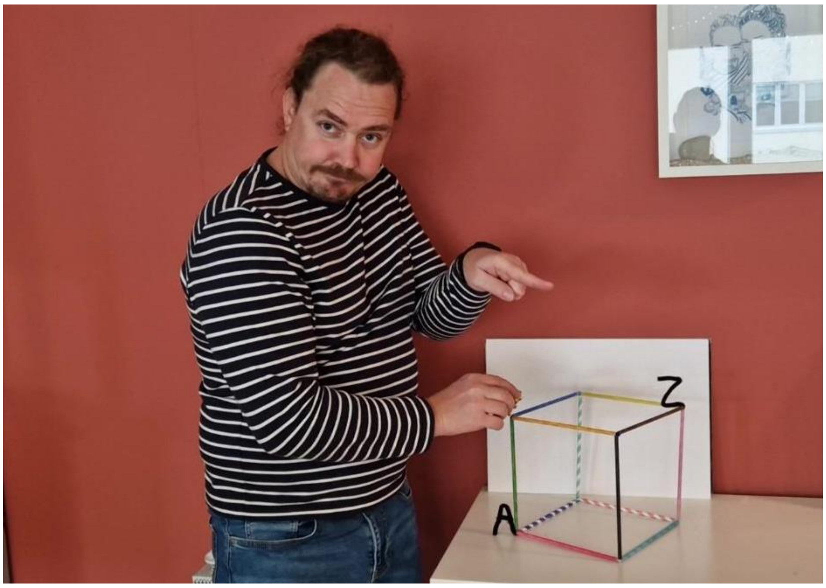


Figure 2: The position of the beetle and the description of the possible paths in DGS

We also present two possible approaches to the task. This corresponds to the idea of the inductive approach proposed by Grote et al. (2018). In Figures 3 and 4, the excerpts from the sign language representations of an example of a favorable and an unfavorable path are linked to the edge model and the corresponding position of the beetle in the model space or in the corners of the cube.

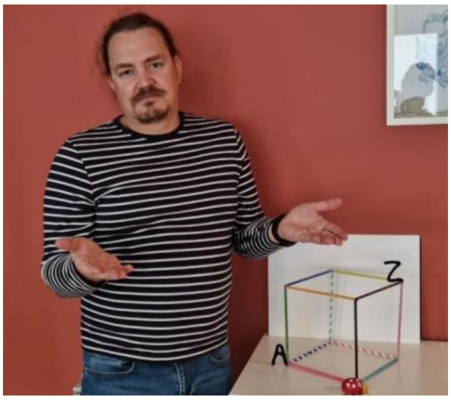


Figure 3: Favorable path "The beetle has reached its destination"

Figure 4: Unfavorable path "Target missed"

With deaf learners in mind, we think it makes sense to provide colored edges in the drawing. In variant A for children who are beginning to learn DGS, a physical edge model is shown, and all sentences are demonstrated on the edge model. In variant B for children who have a good command of DGS, the physical model is not shown but only signed. The picture may be superimposed.

Beetle Task

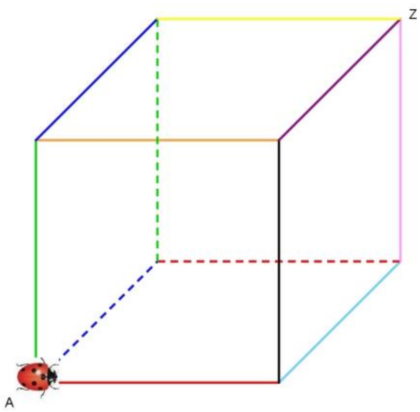


Figure 5: Beetle Task

This is an edge model of a cube. A beetle is sitting in corner A. It wants to crawl to corner Z. The beetle's eyes are blindfolded. The beetle feels its way along the edges of a cube. The beetle stops at each corner and makes a lucky turn. But it never goes back. After three edges, the beetle is tired and stops. In the picture, the beetle first goes along the red edge. If it turns onto the light blue edge at the end of the red edge and onto the pink edge at the end of the light blue edge, it will arrive at Z. However, if it crawls in the order red-black-orange, it will not arrive at Z after three edges.

How many paths are possible? What is the probability that the beetle will reach its destination?

Solution to the Beetle Problem

Task* 1: *Which paths are actually possible?

The learners should not be given any instructions on how to solve this task. In any case, they should be provided with cube models, paper and colored pencils. The colored edges in the template can provide support. Some children will only find some of the paths, others will find all of them.

Solution to task 1:

We draw a tree diagram (Fig. 6, p. 7) that visualizes the beetle's decisions at each corner. At starting point A, it can choose green, blue-dashed or red. If it has chosen green and crawled along this green edge, it is faced with the choice of blue or orange, as it will not crawl back. If it has now crawled along the orange edge, the last choice is purple or black. If it chooses purple, it arrives in Z. Its path can be seen on the third cube from the top. If it chooses black, it misses its destination, as the fourth cube shows.

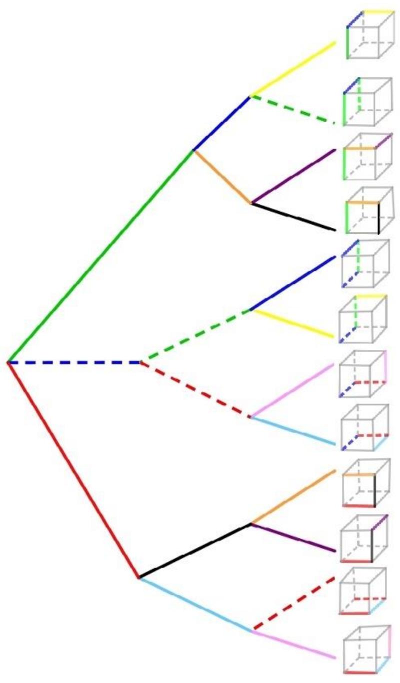


Figure 6: Tree diagram for the Beetle Task

The tree diagram is an important tool in combinatorics and probability theory. It structures/models the real situation and systematically records all cases. Like any visualization tool, a tree diagram must be acquired by learners. Sill & Kurtzmann (2019) write: "To set up a tree diagram, students have to break down the complex process in the task into a sequence of actions or decisions. To do this, it is useful to imagine the realization of a concrete example and ask which actions are to be carried out one after the other [...]. You can orient yourself on the verb used in the task." In our case, it is the sign or the verb for "crawl". At the end of each edge, the beetle decides where to continue crawling.

The colors in the tree diagram correspond to the colors of the edges of the cube. To familiarize yourself with this correspondence, you should trace a path in the tree diagram and at the same time let a (virtual) beetle crawl along the edge model. Conversely, a path should be shown on the edge model in the tree diagram.

The tree diagram as a modeling tool reaches its full potential in secondary school stochastics lessons. It makes sense to introduce learners to tree diagrams as early as primary school when solving combinatorial problems, because: "There are many similarities between the mental actions involved in drawing up tree diagrams in combinatorics and probability theory. This is why working with tree diagrams when solving combinatorial problems is important for further teaching beyond the actual purpose and can be a good basis for working with tree diagrams in probability theory. We therefore recommend encouraging children to set up tree diagrams for all suitable tasks." Sill & Kurtzmann (2019, p. 199)

Some learners may have used a tree diagram to solve task 1, others may have only found individual paths in their own notation. The complete tree diagram should emerge from the individual contributions of the learners in the concluding class discussion. The learners use the finished tree diagram to show "their" paths and, in doing so, demonstrate the unambiguous correspondence between the paths in the tree diagram and the crawling paths of the beetle. You could even draw a large tree diagram with chalk on the tarmac and let the children walk or "crawl" their paths like beetles. The next task can be solved with the help of the tree diagram.

***Task* 2: *Sort all paths into those that lead to Z and those that do not*.**

Solution to task 2:

We can simply use the dice images to count that exactly 6 of the 12 possible paths lead to the destination. To notate and communicate the paths (for the next task), we can use the abbreviation with the colors. For example, $green \rightarrow blue \rightarrow$yellow for the path that leads to the 1st cube from the top.

***Task 3: Why does the path green → blue → yellow have the same chance as the path $red \rightarrow black \rightarrow$ orange*?**

The solution to this task requires a correct understanding of the content of the way of speaking "auf gut Glück". In this context, the way of speaking means that the beetle does not favor any of the possible directions. It is not so easy to make a choice at random without any aids. Think of the game "rock-paper-scissors". An attentive opponent will usually be able to observe patterns in their opponent's behavior and adapt to them. In our case, for example, by throwing a dice with two blue, two red and two green sides – an understanding of the content can be developed or even initiated. If the beetle is in A, then we can "take away" its decision by throwing this dice. The children could carry out and analyze this dice throwing experiment. We rolled the dice 100 times and got the following results:

Color	blue	red	green
Frequency	30	35	35

Table 1: Frequency of the colors, results

These results presented above are completely consistent with the idea of "at random". With only 100 trials, the observed values may deviate considerably from the expected value $100 \cdot \frac{1}{3}$which cannot be realized anyway. This is another realization that is part of a substantive understanding of probability statements: we have to allow chance some leeway. In primary school, we can only achieve a first approximation to this idea. That is why it is important to start with such explorations at an early age and to pick them up again and again.

However, if all edges have the same chance at the beginning in A, then this naturally applies to all subsequent edges for the same reason. No path is favored in terms of chances. Consequently, all paths have the same chance. The sign for chance can be seen in Fig. 7.

Figure 7: The gesture for "probability" or "chance"

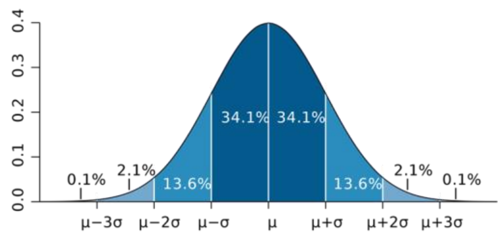
Task* 4: *What is the probability that the beetle will reach its destination?

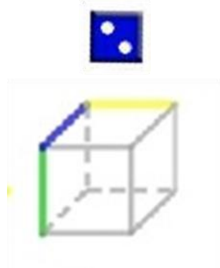
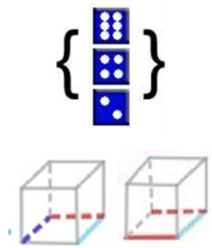
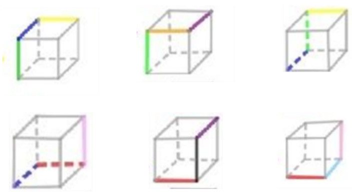
Solution to task 4:

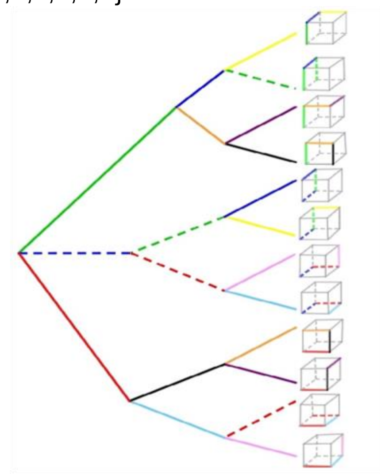
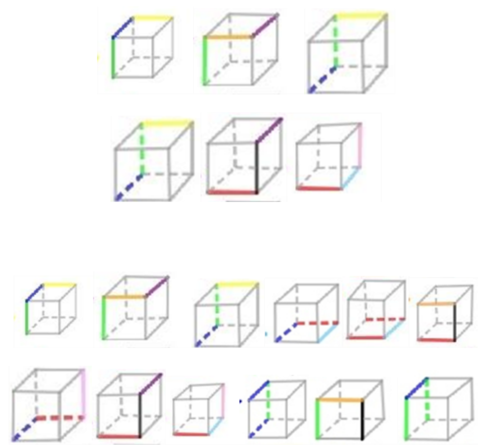
There are 6 *favorable* paths to the goal and a total of 12 possible paths of length 3. All paths are equally probable. A Laplace model is available. The probability that the beetle reaches its destination is $\frac{6}{12} = \frac{1}{2}$. Of course, in primary school we will not talk about Laplace's model, but about the ratio of the favorable paths to all possible paths. Laplace's model is a random experiment with the additional condition that all results have the same probability. To back this up, you can use a *good* coin, which is often used to make a fair decision. There are two possible outcomes: "arms" and "tails". Only one side is favorable for "tails", the coin is likely to fall $\frac{1}{2}$ falls on "tails".

In Winkenbach's lesson experiment (2011), one pupil commented on this result with the words "It's a fifty-fifty, so to speak, a draw!" And another immediately put it into perspective: "But he also got to be lucky!". The last sentence points out that a probability statement does not allow a prediction for the individual experiment. This is also a realization that is part of stochastic thinking and should be introduced early in the learning process.

Mathematical Terms (Stochastics)

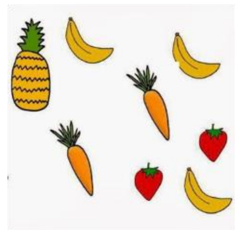
Terminology	Definition	Examples	Illustration
Stochastics	Stochastics is a field of mathematics that deals with the probabilities of events in random experiments.	Stochastics is used as a tool for natural sciences (biology, chemistry, ecology, neuroscience etc.), social science and other sciences.	 Quelle: Probability theory – Wikipedia
Random experiment	Processes that can be repeated any number of times and whose outcome depends on chance.	• Coin toss • Die roll • Blind beetle's route along the edges of a cube	  Quelle: Теория вероятностей: основные формулы, примеры | Калькулятор вероятности онлайн (skysmart.ru)

Outcome	Possible result of a random experiment	• "Number two" when throwing a die • One possible route of the beetle; whether successful or not (for example, green-purple-yellow)	
Event	A set of outcomes	• "All even numbers" when throwing a die • "All ways which lie in the base plane of a cube"	
Favorable event	Outcomes which result in the occurrence of the desired event	• "All successful routes of the beetle"	

Sample space (Ω)	The collection of all possible outcomes is called the <u>sample space</u> of the experiment (	• “All possible numbers” when throwing a die • “All possible routes of the beetle”	{1,2,3,4,5,6}
Tree diagram	A tree diagram is a **graphical representation of the random experiment**. Mimicking the form of a tree, it uses branches to represent the subdivision of possible results or processes.	• “All possible routes of the beetle” are represented through the branches of the tree-diagram • There are 12 possible ways; 6 of them are favorable.	
Probability	The probability of an event is a number between 0 and 1; the larger the probability, the more likely an event is to occur. Laplace model: If all outcomes are *assumed* to be equally probable, then the probability of an event is calculated as the ratio of the number of favorable outcomes for this event to the number of all possible outcomes.	• The probability of the event “beetle will choose successful route” is 50%, since 6 of 12 ways are realized successfully. $P(beetle'ssuccess) = \frac{6}{12}$	

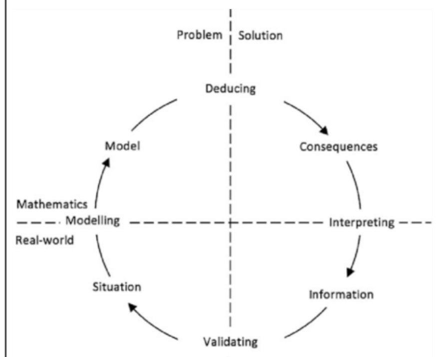
<table>
<tr><td>Frequency</td><td>How often something (outcome, event) occurred in a statistical experiment</td><td>8 children were asked what they would like to eat

● 3 like bananas
● 1 like pineapples
● 2 like carrots
● 2 like strawberries

The frequency of children who like bananas is 3.</td><td rowspan="2">
There are 8 possible choices. 3 of them are bananas.

Quelle: https://i.pinimg.com/originals/01/d7/32/01d7326f42eefb0aaf0d45f5dbb3ab22.jpg</td></tr>
<tr><td>Relative frequency</td><td>The relation between how often something occurs in relation to frequency of all outcomes</td><td>● 3 of 8 children like bananas. The relative frequency of children who like banana is 3/8.</td></tr>
</table>

Didactical Terms (stochastics)

Terminology	Definition	Examples	Illustration
Experiment	Once the stochastic model is chosen the random experiment can be conducted.	• The learners can throw a die three times to model the potential route of the beetle.	green-purple-yellow 
Stochastic Modelling	Stochastic models are mathematical model which represent or describe real processes or part of it and must be distinguished from reality itself.	• Dice, coins, or other "gambling equipment" which are known to learners from their everyday context can be deemed a part of the real-world. When a die is referred to as a "random device", this should mean an ideal die in which all sides have the same probability and exactly one side is always on top when thrown. Such an ideal cube or coin can also be used for Laplace experiments. • In our teaching example we choose dice with colored sides to model the possible ways of the beetle.	 Schupp 1989
Analyze data	Data can be documented in tables, charts or tree diagrams	• All possible routes can be documented with sketches of cubes with potential routes	See table above

Structure recognition	Recognise the structure in the random experiment	• Possible outcomes of the experiment can be structured with the help of tree diagrams	See table above
Calculations	Estimate frequencies and probabilities of events	• Frequencies can be calculated and compared with theoretical probabilities	See table above

6. Sign Geometry

Nordheimer, S. & Sell, T.

Introduction

Considering Polya's (1969) theory of mathematical problem-solving, which did not lose its relevance for didactics of geometry and is still foundational today (Weigand et al., 2018), stereometry plays a decisive role throughout one's entire academic career. According to Frick (2019): "The present findings point to a tight connection between early mental transformation skills, particularly the ones requiring a high level of spatial flexibility and a strong sense for spatial magnitudes, and children's mathematics performance at the beginning of their school career." Although Signed Geometry seems to have positive effects on visual-spatial abilities of all learners (Groninger & Sieprath, 2019) this paper focuses on deaf and hard-of-hearing learners whose talents can be overlooked by educational researchers and teachers (Weber et. al., 2023). When discussing teaching Signed Geometry to deaf and hard-of-hearing learners, we will first consider the theoretical framework of the mathematical abilities of deaf learners established and used for empirical studies by Rosanova (1991).

After that, we will refer to empirical findings which deal with geometrical learning and teaching of deaf learners on one side and visualization of mathematical content on the other. We will then use these contributions as building blocks for the theoretical framework (Niss, 2019) of Signed Geometry and derive ideas for further theoretical and empirical studies, as well as didactical consequences for teaching Signed Geometry at school. We then introduce concrete teaching examples as parts of the empirical data we gathered through cooperation with deaf teachers and the creation of teaching materials for deaf learners in Bonn Math Club to illustrate the theoretical aspects. The presented example was introduced to the deaf learners and their teachers as a part of the project *Signed Mathematical Challenges* in the beginning of 2023. The aim of the given problem is to provide concrete ideas for teaching, to illustrate theoretical considerations synthesized from different researchers, to apply them into school praxis and to reflect theoretical and empirical suggestions in cooperation with school teachers.

Theoretical Background

To describe mathematical abilities and potentials of typically hearing school children, Krutetskii (1976) combined cross-sectional and longitudinal examinations and used a precisely designed series of mathematical problems. Due to the size of the research populations examined in the studies, Krutetskii's work remains unique in the field of mathematical education and research focused on mathematical creativity and giftedness in general (Leikin, 2021).

Krutetskii (1976) differentiated among three distinct types of mathematical abilities on different levels:

- **Analytic** – very strong verbal–logical component predominating over a weak visual–pictorial component; spatial concepts weak; cannot use visual supports in problem solving and feels no need to use the visual support.
- **Geometric** – very strong visual–imaginative component, predominating over an above average verbal–logical component; spatial concepts very good; can use visual support in problem solving and feels a need.
- **Harmonic** – strong verbal–logical and strong visual–imaginative components in equilibrium; spatial concepts good. Subtype (a) (abstract harmonic) – can use visual support in problem solving but prefers not to. Subtype (b) (pictorial harmonic) – can use visual supports in problem solving and prefers to do so.

However, Krutetskii was criticized by Kolmogorov (2001) for not taking into account the possibility of the special needs of learners who showed outstanding mathematical talents and performance. Deaf learners, with a linguistic repertoire which includes signed languages, were not considered either in Krutetskii's, nor in Kolmogorov's works. Following Krutetskii's understanding of mathematical abilities in general, Rosanova (1991) studied the development of mathematical abilities of deaf learners in school environments. She would show that Krutetskii's typology is applicable not only for typically hearing students but to deaf learners as well. According to Rosanova (1991), deaf learners who belonged to the group with strong verbal-logical and visual-imaginative reasoning showed the best performance in school mathematics.

Presmeg (1986) followed Krutetskii's research and focused on the connection between mathematical giftedness and visualization. She asked for reasons why the so-called non-visualizers - who belonged to the first group according to Krutetskii's approach - were often more successful in school mathematics than visualizers. Besides external factors rooted in the teaching methods and learning environments - which gave more room and appreciation to non-visual and analytical ways to solve problems in classical mathematical classroom settings - she was searching for internal factors for the success of non-visualizers. In her interviews with hearing high school students, she would see that non-visualizers often used general non-visualized formulae to solve mathematical problems more rapidly. A secondary reason which could prevent visualizers from successful problem solving when compared to non-visualizers could be the challenge to overcome one-case concreteness of the visual image to find general solutions to mathematical problems. According to Krutetskii, to be helpful in mathematical thinking and problem solving, the visual imaginary has to be controllable and generalizable by the problem solver:

*The fact is that the graphic schemes used by these pupils [i.e., visualizers of high ability] are a unique synthesis of concrete and abstract. The 'geometer' pupils feel the need to interpret a problem on a general plane, but for them this general plane is still supported by such images. In this they differ from pupils of little ability - for whom visual images really bind thinking, push it onto a concrete plane, and hinder the interpretation of a problem in general form (*Krutetskii *1976, pp. 325-326).*

In line with Krutetskii's and Presmeg's ideas for hearing students, Rosanova (1991) suggests that to develop mathematical potentials and abilities of deaf learners, it may be necessary to pay more attention to the generalization processes. We suggest giving deaf students not only opportunities to create visual images but also to find ways for the controlled use of geometrical visualizations. In geometry classes, this can be achieved in two ways. Firstly, it may be helpful to make explicit to the learners when the image or visualization represents one concrete example or when its aim is to visualize general propositions.

To communicate about the scope of the geometrical visualization, **conventionalized** and **productive signs** and gestures could be used as instruments for the controlled use of geometrical images to solve mathematical problems (Nordheimer et. al., 2024). Secondly, variations of geometrical visualizations and the building of **geometrical patterns** from many different cases can be helpful to teach deaf learners how to generalize geometrical visualizations and to derive general propositions by studying many cases and comparing them with each other (see also Presmeg ,1986). Later in this paper, an example will be given showing how this could be achieved in geometry lessons when referring to the volume of a cube. Rosanova (1991) suggests when teaching deaf learners to pay more attention to the development of **verbal-logical** and so-called **visual-imaginative** thinking as an interplay of components. We aim to go further and to find ways to foster verbal-logical and visual-imaginative thinking by careful and conscious targeted implementation of Signed Geometry and signed languages into mathematical learning and teaching processes by giving concrete geometrical examples. To do so, we will first look at some empirical findings concerning the geometrical thinking of deaf children.

Empricial Findings

Empirical studies focused on the teaching of geometry to deaf learners stress the connection between learners' abilities to solve geometrical (and especially spatial) problems and their general mathematical abilities. Not all studies actively contribute to the value of signed languages in the development of geometrical skills, but there are empirical results which support the relevance of teaching geometry in signed languages. We will present some relevant empirical findings here.

Geometry and spatial thinking of deaf learners

Zarfaty et al. (2004) studied 3- and 4-year-old deaf children's ability to remember and to reproduce the number of colored bricks in a set of objects and suggests that deaf children benefit from mathematical teaching methods that emphasize the spatial representation of numbers. On the other hand, Chen (2022) found a strong correlation between spatial ability and the mathematical performance of deaf learners by asking 256 deaf schoolchildren in Grades 3 to 9 in two special education schools in China to perform cognitive and mathematical tests. Based on another empirical study with 198 deaf and hard of hearing students, Chen and Wang (2020) even suggest that mathematical achievement of deaf learners depends more strongly on spatial ability than on specific numerical abilities. However, these findings may depend on the cultural and educational context of the teaching of mathematics and on specific educational traditions and systems in China which differ from those employed in European contexts.

The results of the research of Marschark et al. (2015) also suggest that there is more empirical evidence proving an advantage for deaf students in the spatial domain than in the visual domain. This fact leads us to the assumption that it may be important to expand the framework of the mathematical curriculum regarding stereometry and not to place such a heavy emphasis on plane geometry as is the current state of affairs expressed in the traditional curriculum. Marschark et al. (2015) also found that the "performance of deaf and hearing individuals on the same visual-spatial tasks was associated with differing cognitive abilities, suggesting that different cognitive processes may be involved in visual-spatial processing in these groups" (Marschark et al., 2015).

Relative strength of deaf learners in geometry

Pagliaro and Kritzer (2013) examined the performance of deaf and hard-of-hearing children, 3–6 years of age, against a developmental trajectory of early mathematics concepts and skills. The results of these studies show "shape" and "geometry" as areas of strength relative to other areas. These results are in line with later research conducted by Wauters et al. (2023) that also found strengths in the geometry domain and challenges in the area of measurement in deaf and hard-of-hearing children. The role of sign languages in the mathematical development of DHH children was not considered in these studies.

Using sign language is critical for deaf children to develop spatial and geometrical thinking

To describe the role of signs and gestures in the process of geometrical problem-solving, we would like to refer to the work of Johnson (1987 cited in Campbell et al., 1995), who differentiates between abstract propositional structures, image schemata and particular concrete "rich images." As an illustration, we will provide an example given by Campbell et al., (1995). The concept of the triangle and its properties is a part of an abstract propositional structure. In contrast, a "rich image" is a particular picture of one specific triangle in the mind of a problem solver. Image schemata build the bridge between abstract ideas and concrete images. This kind of image use can be considered as an important step of schematization and generalization of geometrical visualizations. Productive and conventionalized signs support the generation of image schemata and have an impact on the perception of geometrical shapes.

The views described above could be supported by older studies conducted by Dyachkov (1961), who worked with deaf children and young people without the experience of being educated in orally-oriented schools. Dyachkov's study demonstrates the great role of signed languages in the development of the perception of geometrical figures and, especially, solids. As evidenced by the study, 7 and 8-year-old children who did not know signs for the shapes had difficulties with visually distinguishing geometrical figures and solids. Children who possessed signed designations picked up objects 2 - 3 times more accurately. At the same time, the degree of shape distinction directly depended on the degree of knowledge of signs. Dyachkov worked with children and young people who, for various reasons, were unable to attend school and were therefore not specifically supported with spoken language. He discovered that children of deaf parents were more successful in differentiating geometric figures and solids.

Leaning on Dyachkov's findings, Suchova (1966) carried out various long-term studies of geometry teaching in several schools for deaf children. Based on this, Suchova suggested connecting geometry with real-life instructional scenarios (for example working with wood and producing objects of everyday life). She also recommended using geometry as a teaching and demonstration tool to instruct students in other areas of mathematics like arithmetic. In our teaching example below, we show how cubes as geometrical objects can help learners visualize cubic numbers like 64. In the chapter focused on stochastics (Warmuth et al., 2025), you can find out how geometry can be used to teach stochastics.

To study the effect of signed languages and specific experiences of perception on the spatial abilities of deaf children Parasnis et al. (1996) compared deaf and hearing children. She could not find a significant difference in their performance on the visual spatial skills tests, suggesting that deafness per se may not be a sufficient factor for the enhancement of visual spatial cognition. In agreement with Dyachkov, she found that exposure to sign language and fluent sign skills may be the critical factors that lead to differential development of visual spatial skills in deaf learners.

Based on empirical results Emmorey (2023) claims that experiences with sign languages can enhance mental rotation ability. That could be explained with the fact that comprehending spatial descriptions from the signer's perspective requires a mental transformation of locations in signing space (Secora & Emmorey, 2020).

Hands-on approaches to solving visual-spatial problems

Yashkova (1988) aimed to describe deaf children's thinking processes when performing practical problems of a visual-spatial character. The operations of analysis and synthesis, inseparably connected with each other in the process of any mental activity, were of great importance in solving these problems. The peculiarity of the tasks consisted in the fact that their performance required the ability to switch from object-action forms of analysis and synthesis to mental ones and then to switch back. To support the mental analysis of said objects without possibility of action with them, some participants used signs or gestures which represented actions and operations within an imagined apron.

Sture (1984) studied how deaf learners solved problems in physics in comparison to hearing students using problems with spatial components. The problems needed to be understandable to deaf learners and at the same time require from them an active analytical approach revealing essential connections. These requirements were met to a certain extent by setting a specific practical experimental task before the pupils. In accordance with the instruction, the pupils were to make a ball fall from the inclined plane into each of the four compartments of the box in turn and explain the results obtained. Many deaf learners, even before starting the experiments, gave the general correct solution: they offered to raise or lower the chute depending on the tasks set. The further content of their actions consisted in "trying on" the height of the launching point to the necessary range of flight. This was achieved most often during practical trials, but often students mentally traced the path of the ball, accompanying its "movement" with imitating signs or gestures, and correlated the predicted range of flight of the body with the height of its fall. The use of signs or gestures in problem solving was observed more often in the deaf students as opposed to their hearing counterparts. As a result of such actions, many deaf students were able to get the balls into the desired range interval with sufficiently high accuracy. Their predictions were often more accurate than those of the hearing students.

Rosanova's (1978) study showed that deaf children used words less often than hearing children to memorize geometrical figures or solve visually presented problems. They used more signs or gestures than hearing pupils to support memorization and problem-solving. Based on these studies with deaf children, Rosanova claimed that language and thinking form a unit, but they are not identical. It is therefore crucial for the development of thinking that children learn to operate with mental images. Their images of objects and ideas should become increasingly generalized and detached from concrete objects and actions with them. Rosanova assumed that actions, motoric body movements, models, pictures, gestures, signed languages or other language systems and modalities regulate thinking and help them to operate with mental images. According to this view, the best way to develop geometric and spatial thinking is to allow operations with mental images by solving problems which could be represented in actions, pictures, models, signs or words in the beginning and to move toward operating with mental images using signs, gestures or words without using tangible or visible objects.

Hints that deaf children are natural visual learners?

Presmeg (1986) described visual imagery as a continuum from concrete to abstract and suggested placing Johnson's "rich images" at one end of the continuum and his "image schemata" at the other. In an earlier study, Presmeg (1986) proposed that there are two types of visual images involved in different situations that have different impacts on the mathematical performance of students. For example, Pitta-Pantazi and Christou (2010) suggest for elementary students that one type of visual image is more figurative, skeletal, symbolic and generic, and the other type is more concrete, pictorial and colorful. While students with high achievement in mathematics operate with skeletal, figural or schematic images, students which are not so successful in mathematics tend to produce more detailed, colorful and pictorial images. This distinction is similar to the categories "schematic" and "pictorial" used by Blatto-Vallee et al. (2007) to investigate visual-spatial representations in mathematical problem solving by deaf students. Comparing the results of deaf students to those of their hearing peers, Blatto-Vallee et al. (2007) conclude that deaf students use more pictorial than schematic representation to solve mathematical problems. In this study, hearing students used more "schematic" representations and outperformed their deaf peers. A closer look at the study reveals that the items of the study included 15 mathematical word problems without mathematical symbols, geometrical drawings or sketches. Sign languages or gestures are not considered in the study. For this reason, the study doesn't provide insights into the visual-spatial abilities of deaf learners. But it does provide empirical proof that word problems alone are not sufficient to investigate the visual-spatial skills of deaf learners and gives us additional arguments for multi-modal teaching methods like those suggested by Skyer (2023).

Deaf students are often considered visual learners who profit from the visualization of mathematical teaching materials with “language poor” or even “language free” learning environments. However, the results of the investigations by Marschark et al. (2015) showed that performance on the Spatial Relations task was related to the deaf participants’ language ability in their preferred modality (sign or spoken language). Using and fostering the preferred mode of communication and instruction appears, in this case, to be more relevant than focusing on the specific visualization of mathematical ideas. To strongly emphasize visualization without targeted language support and teaching in geometry lessons could presumably lead to children relying too heavily on pictorial and concrete images and prevent them from developing schematic images of abstract concepts and ideas. We will now summarize the didactical consequences of our theoretical contributions supported by empirical research.

Didactical Consequences

Before we give some practical examples for teaching, we would like to summarize the results of the theoretical findings of the studies mentioned which are relevant for teaching geometry to children who are deaf and hard of hearing.

- Geometry seems to be an area of strength for deaf children and can be used to teach other mathematical areas like arithmetic.
- To teach geometry, it is important to connect active operations with models and visualizations embedded in the language to help students to produce not only concrete pictorial but also schematic images of the abstract geometrical concepts.
- Sign languages are not only the preferred mode of communication for many deaf children, but also the tool which helps them to perceive objects, to memorize concepts and to solve problems.
- Spatial geometry seems to play a crucial role for mathematical development in children.

To give some concrete ideas as to how spatial geometry and arithmetic can be connected by use of signed languages and 3D models, we will now concentrate on the idea of the volume of a cube as follows below. We would like to make some suggestions for mathematics lessons starting in 3rd grade. The ideas can be adjusted to the lessons with older students by implementing fractions and even talking about the possibility of cubes with irrational volumes. We entrust teachers as pedagogical experts with the testing of our suggestions in the school setting and welcome feedback.

Teaching Example

In our first section, we saw how important it is to teach spatial geometry to deaf children. It is of great significance that sign language connects the models and visualizations from the very beginning and contributes to the development of concrete, but also schematic images. Operating with schematic images can contribute to the development of abstract ideas. An example of such an idea is the volume of a cube. There are various approaches that may be applied to the subject. For example, teachers could cut or saw cubes of plasticine, cheese, soap or even wood into smaller cubes together with the learners in craft lessons. It is crucial that the actions are introduced by signs and gestures derived from the actions on the one hand and documented by videos and pictures on the other.

As an introduction, for example, the origin of the lexical sign that corresponds to the word "cube" can be discussed with the children. The children can be given a ready-made cube and experiment with it or be given the task of making their own cube, e.g. from soap or potato. The aim should be to create a fair die where every number is equally likely. By analyzing the gesture and realizing it through their own actions, they can discover that a fair cube cannot be a cuboid, for example, but must be a cube in the geometric sense. Its edges must be of the same length and its sides must be of the same size.

Once the children experimented with the lexical form and produced their own cube, they can be exposed to a productive description of the cube where two flat hands substitute parallel surfaces of the cube. The children can look at the teacher’s descriptions and repeat them by forming with their hands all six surfaces. Once they are familiar with the productive description of the cube, they can try to describe the height, length, and width of one particular cube presented by the teacher. They can move toward the generalization of the image by varying the side length of the cube and by describing the length of the edges of their own cube and then by doing it for the cubes of their classmates.

The exercises can be documented as videos or photos which can be enriched with geometrical sketches not only by the teachers but also by the learners themselves. These variations and documentations would help the learners to move from concrete examples to schematic representations mediated by conventionalized and productive signs. As demonstrated by the teacher in the picture, they can first draw the vertices, which form three right angles and can be interpreted as a part of the Cartesian system of coordinates. In this concrete example each vertice is 4cm long which is shown by the teacher and can be transferred to their own cubes by varying the length.

Results with practical Relevance

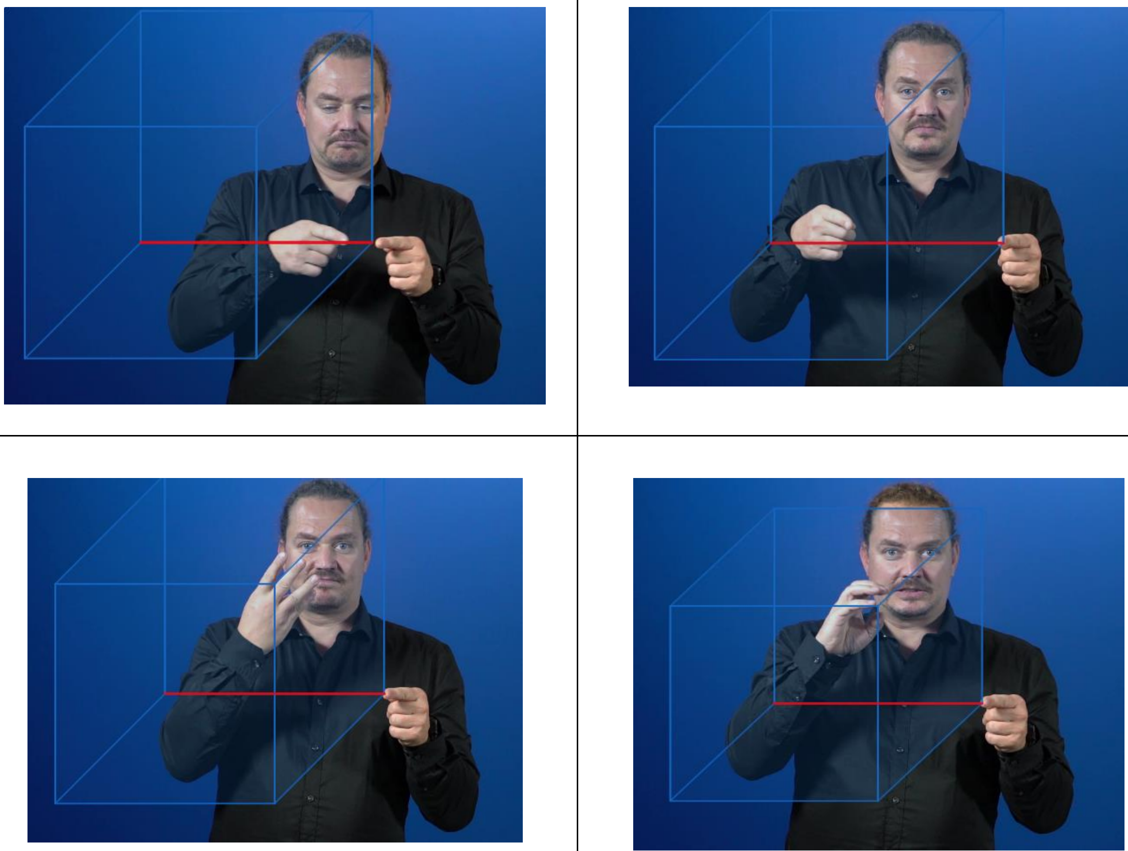


Figure 1: The length of the cube is 4 cm

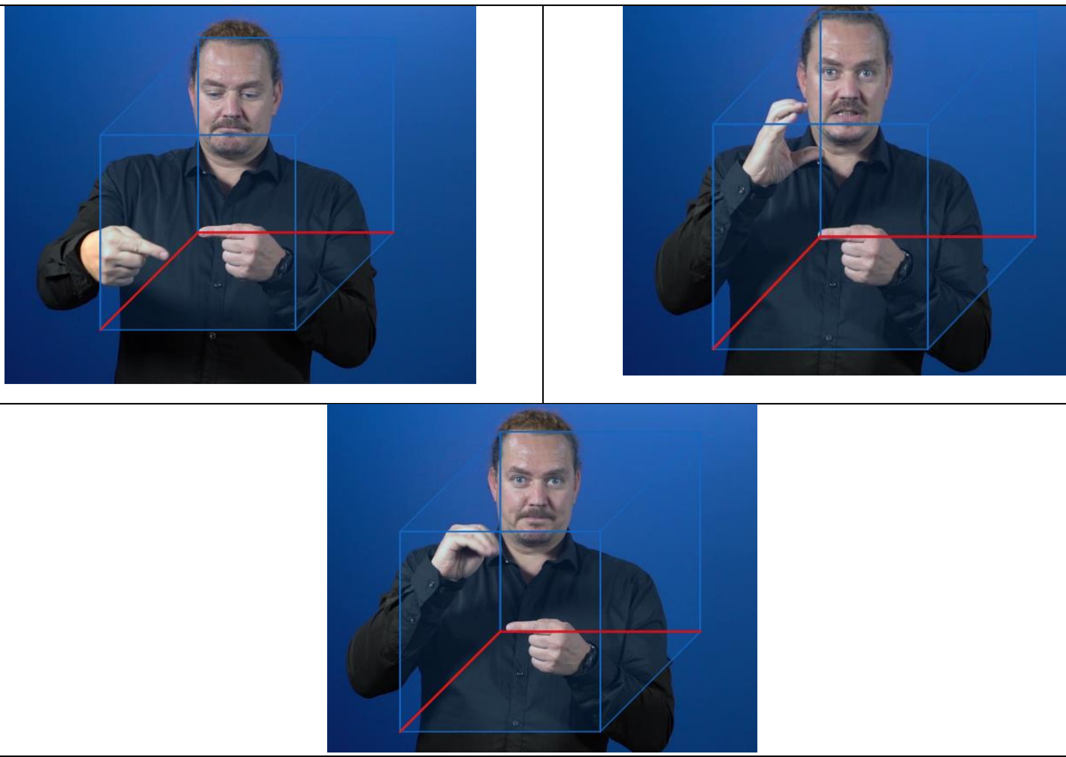


Figure 2: The width of the cube is 4 cm (excerpts)

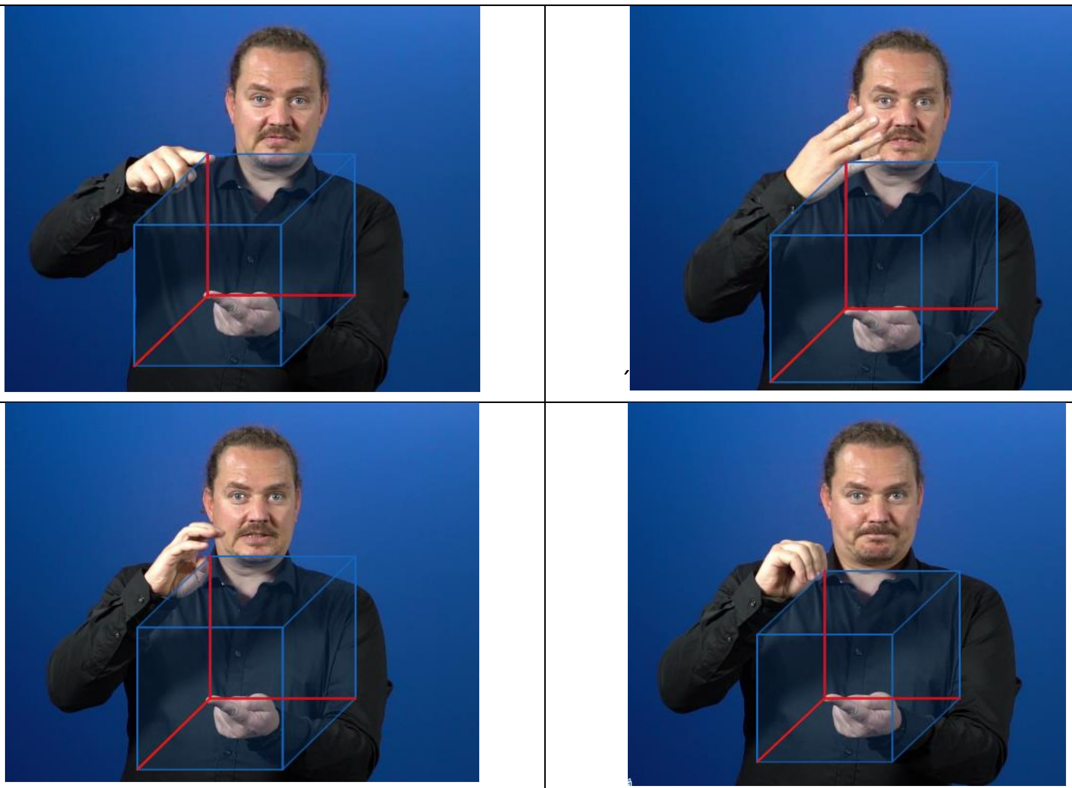


Figure 3: The height of the cube is 4 cm (excerpts)

In the next step, the cube, which is invisible in the video and made visible with help of transparent images produced by GeoGebra, will be divided by parallel lines into 64 pieces. The children can watch the video without sketches or study the series with pictures which connect signs and sketches of geometrical solids. The aim is that they can recognize the actions they completed with soap or wooden cubes. They can try to count smaller invisible cubes or go back to the model cubes to control their counting. The actions, the videos and the sketches connected through productive and conventionalized signs represent concepts by giving the students the possibilities to build concrete pictures of a bigger cube which is divided into smaller cubes which also represent cubic number or operation 4x4x4. But it also opens various possibilities to move forward schematic images of cubes and to grasp the abstract idea of the cube as a solid with edges of the same length and surfaces where two of them are parallel to each other. These are properties which can be generalized.

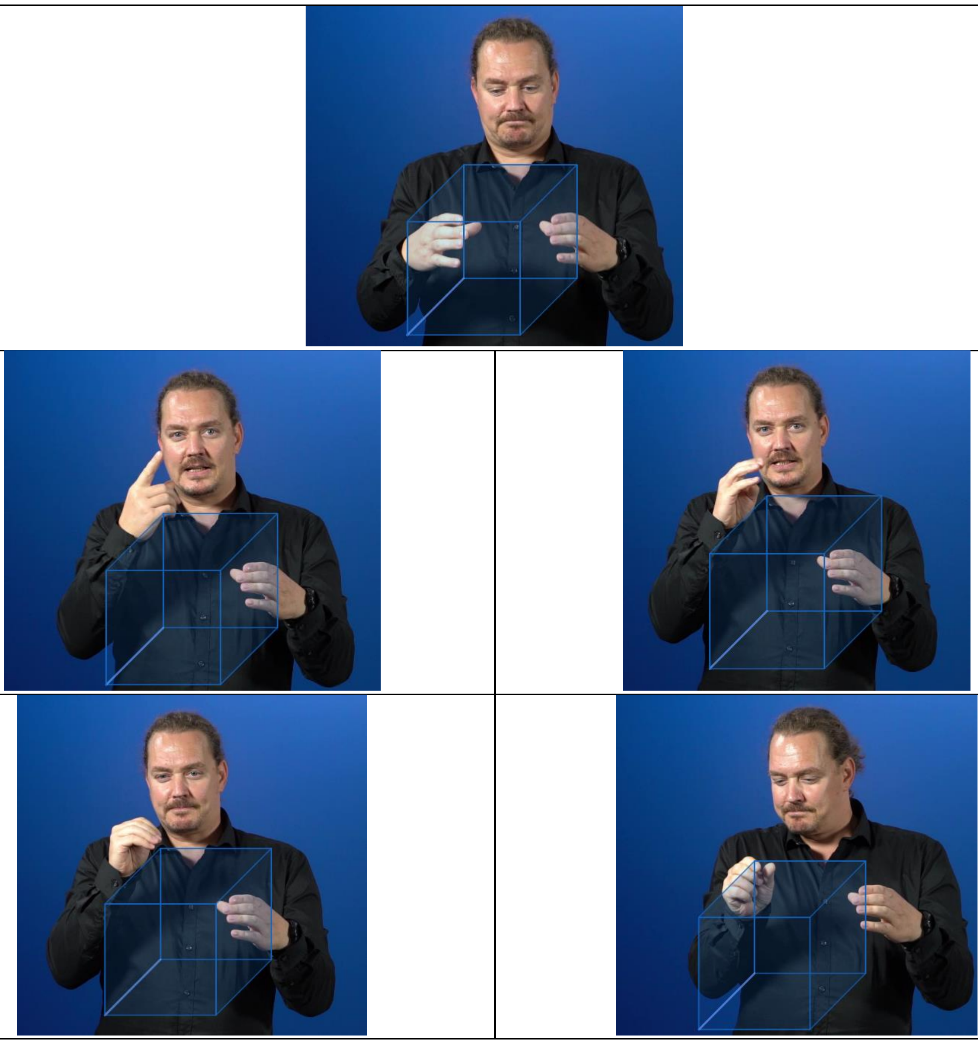


Figure 4: The distance between the cutting planes is 1 cm.

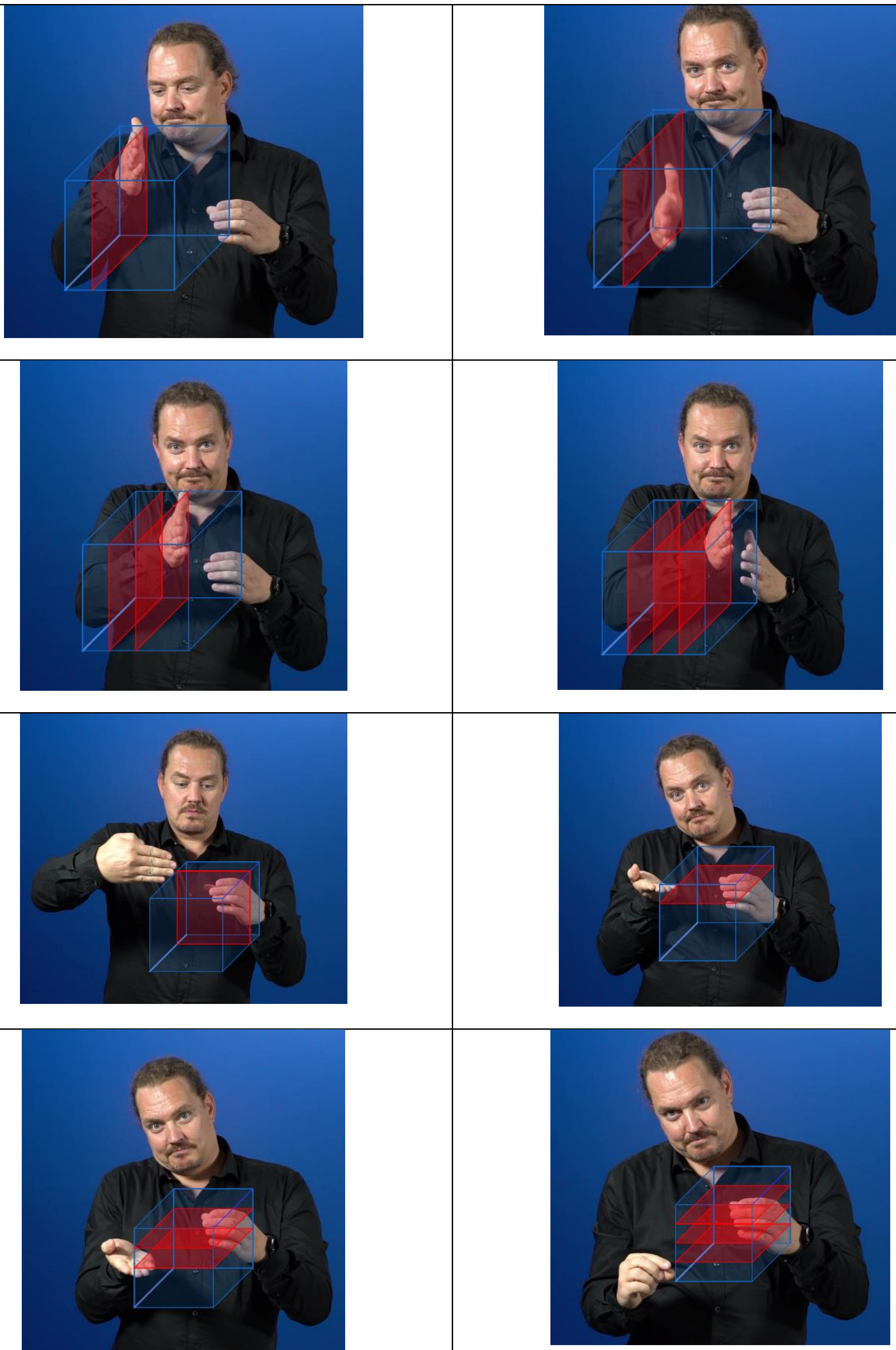


Figure 5: Cutting of a cube by parallel planes.

By cutting different cubes in a similar fashion or the same cube into smaller cubes as a next step of exercises, students can investigate the concepts of volume and other units of measurement by imagining cubic units whose side length strives to zero, while preparing for the complex idea of the volume of geometrical solids by transcending visible models from the real world.

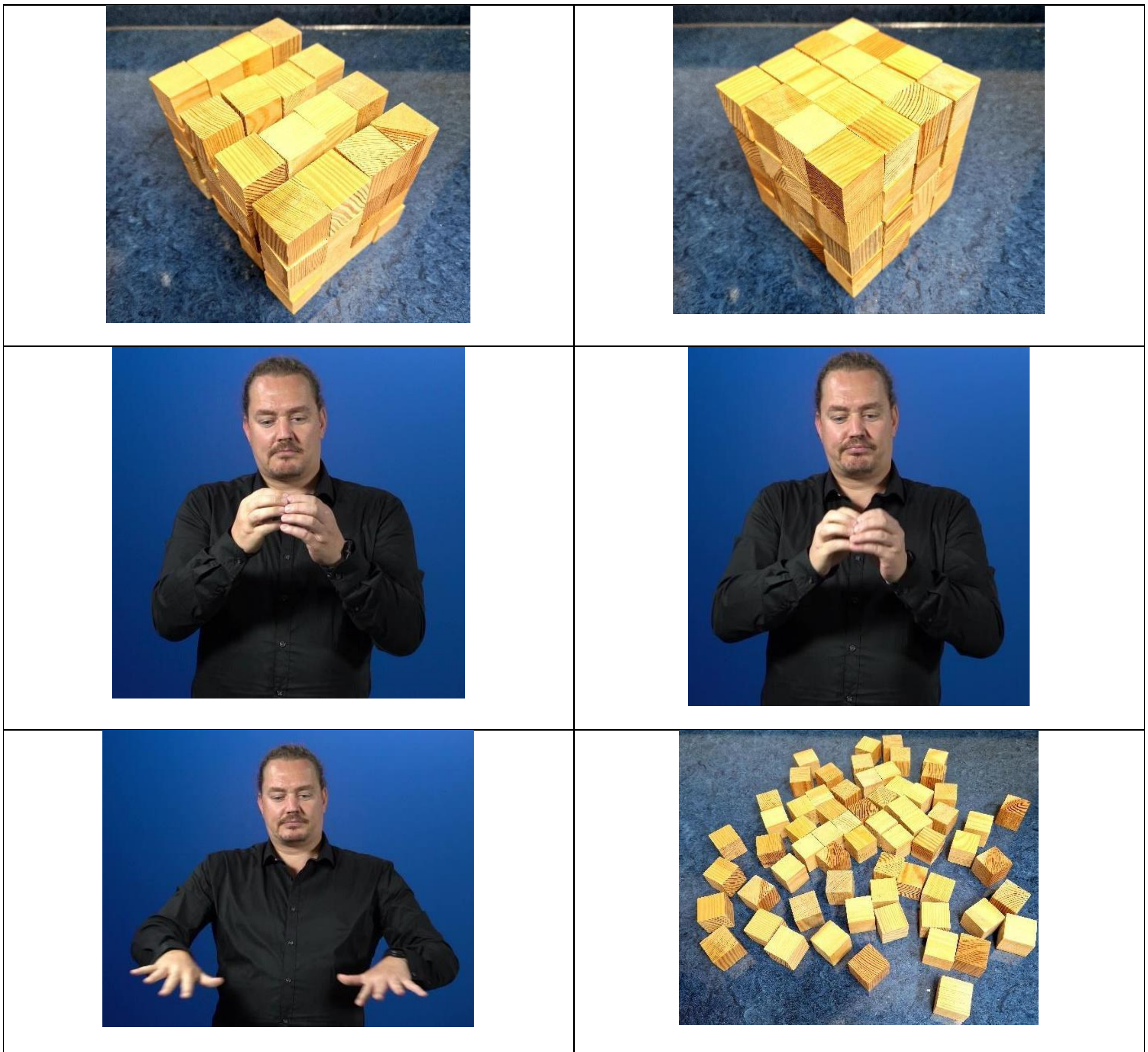


Figure 6: Transformation of cube in German Sign Language and its wooden model.

As the cube and its division through parallel planes is precisely described, the main question of the problem can be posed in German Sign Language: How many unit cubes are there in the bigger cube?

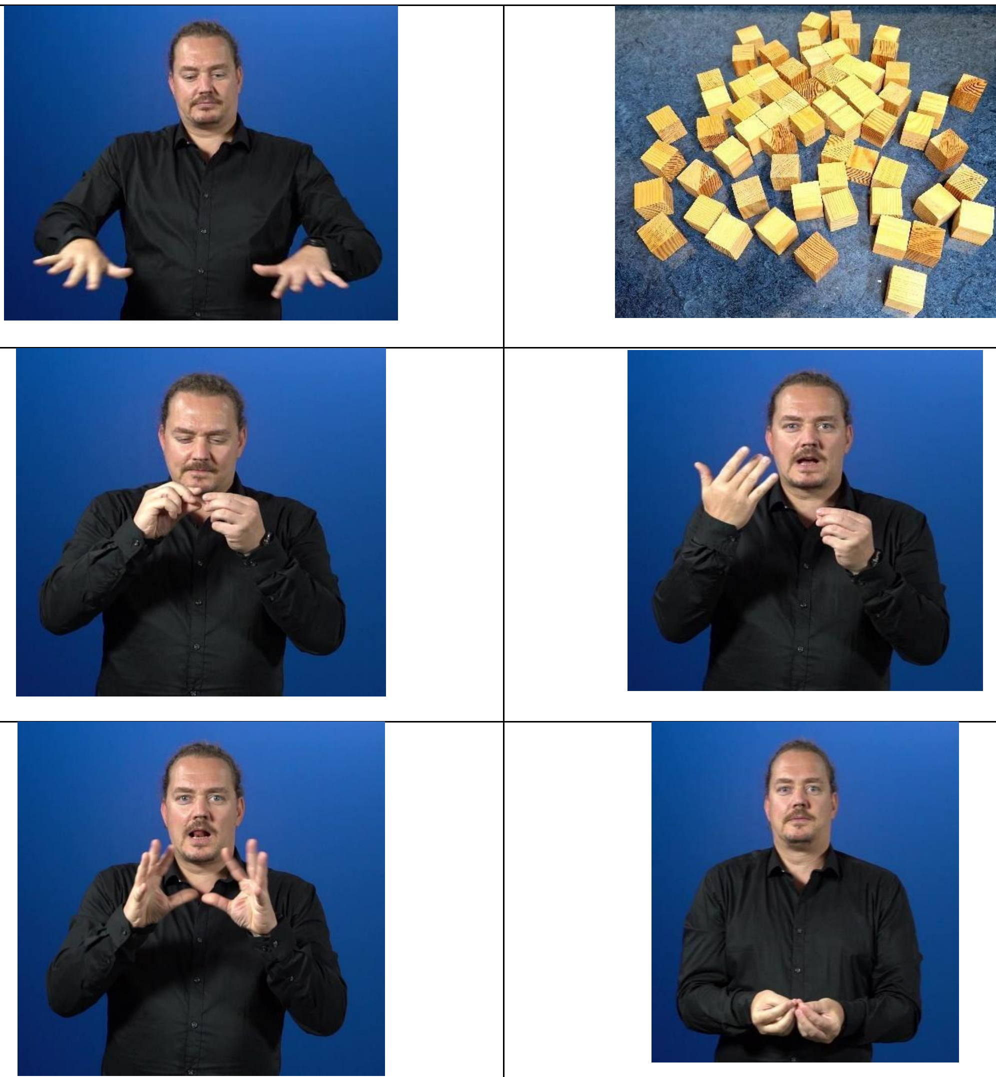


Figure 7: How many unit cubes are there in the bigger cube?

To conclude our contribution, we will now look back again and make explicit connections between our theoretical ideas and the concrete examples provided above:

- Geometry seems to be an area of strength for deaf children and can be used to teach other mathematical areas like arithmetic.

  We used the volume of a cube to visualize a concrete cubic number which can be interpreted as a result of three times multiplication of the same number. The number can be interpreted as the amount of cubes in the height, length and width of the cube.

- To teach geometry, it is important to connect active operations with models and visualizations embedded in the language to help students to produce not only concrete pictorial but also schematic images of the abstract geometrical concepts.

  We operated with an invisible cube by cutting it in smaller cubes and connected it with a wooden model of the same shape.

- Sign languages are not only the preferred mode of communication for many deaf children, but also the tool which helps them to perceive objects, to memorize concepts and to solve problems.

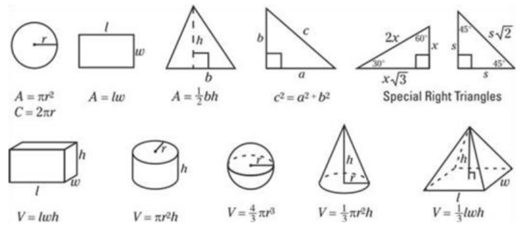
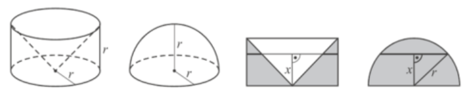
  We gave a representation of the problem in German Sign Language and gave the students the possibility not only to see the cube in the wooden model or in the visually perceived signs but also to feel it in their hands by repeating the signs themselves.

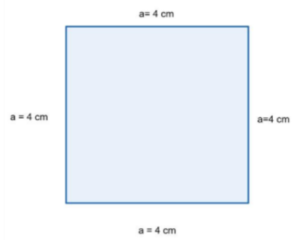
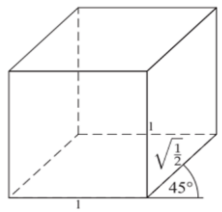
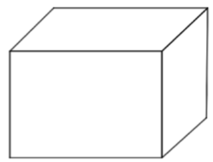
- Spatial geometry seems to play a crucial role for the mathematical development of children.

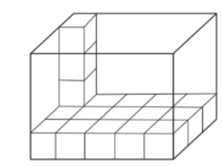
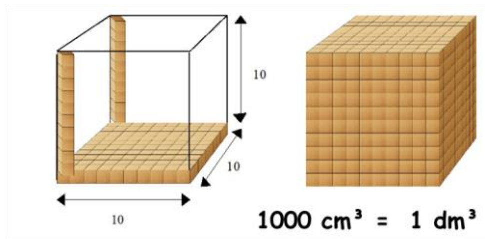
  We used a concrete cube with a volume of 64 cm^2 to introduce the idea of volume and gave some ideas as to how the presented problem can be varied by the deaf learners to generalize the concrete created spatial image.

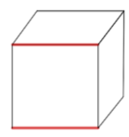
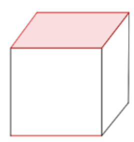
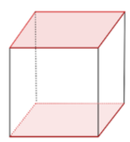
To sum up, we encourage practitioners to present our signed problems to learners, to find their own ways to describe different cubes with different volumes, to move further to other geometrical solids and phenomena. We look forward to teacher feedback and to constructive criticism – a process which is difficult to overemphasize when it comes to development of educational science and practice.

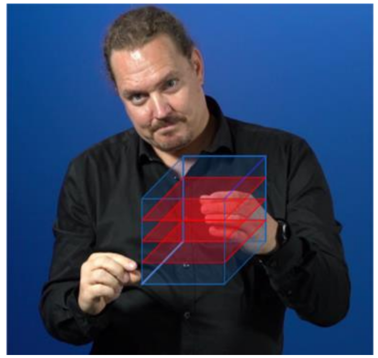
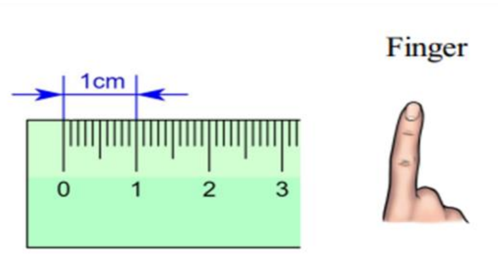
Mathematical Terms (Geometry)

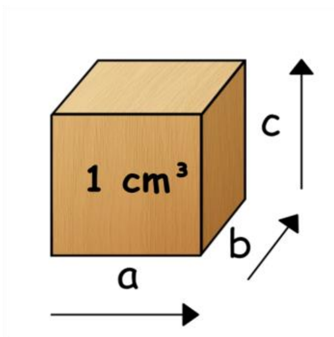
	Terminology	Definition	Examples	Illustration
1	Geometry	**Geometry** is a mathematical field concerned with properties of space such as the distance, shape, size, and relative position of figures. **Solid geometry or stereometry deals with solid figures.** A **solid figure** is the region of 3D space bounded by a two-dimensional surfaces. *Vincenzo De Risi (2015). Mathematizing Space: The Objects of Geometry from Antiquity to the Early Modern Age. Birkhäuser. pp. 1–. ISBN 978-3-319-12102-4. Archived*	Solids like, for example, cuboids, cubes, pyramids, cones, and spheres are defined by their volume and surface area, as well as other properties, and plane geometric figures are defined by their perimeter and area. These concepts belong to important mathematical content subjects and contexts to train mathematical problem-solving. For example, a solid **ball** consists of a sphere and its interior.	 Meeting Numbers Head-On: The SAT Math Tests - dummies
2	Geometrical proof	A geometric proof is a deductive reasoning reached using known facts like Axioms, Postulates, Lemmas, etc. with a series of logical statements. In school geometry, visual proofs and their descriptions can be used as a less formal method of geometrical argumentation and preparation for more precise mathematical methods and scientific work.	For example, to prove the formula for the volume of the interior of the sphere, it can be first shown that the volume of the hemisphere is equal to the volume of the difference between the cylinder and the cone.	 09_Koerpergeometrie.pdf (hu-berlin.de) (Filler)

3	Square	A plane figure with four equal straight sides and four right angles. A square is a special rectangle with equal sides.	The cube from our teaching example has 6 squares. Every square has 4 equal sides. The length of each side is 4 cm.	
4	Cube .	A geometric solid bound by six squares. Each cube has 8 vertices and 12 edges, which are all the same length. Cubes are regular polyhedra (Platonic solids) and are also known as hexahedra.	In our teaching example, we use a problem with a cube with an edge length of 4 cm.	 09_Koerpergeometrie.pdf (hu-berlin.de) (Filler)
5	Cuboid	A geometric solid which has six rectangular faces at right angles to each other. A cube is a special cuboid.	In the next example, we will use a cuboid with following measurements: Length = 5 cm Height = 4 cm Width = 3 cm	

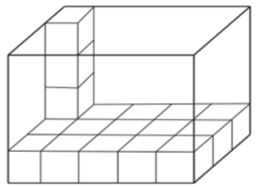
6	Volume (cube and cuboid)	Volume is a measure of the space occupied by a solid. The volume of a solid is determined by its shape and linear dimensions. The main property of volume is additivity, i.e. the volume of any body is equal to the sum of its volumes of its non-intersecting parts.	For example, the volume of the cuboid presented above can be measured by the number of unit cubes. To introduce the idea to learners, a transparent box can be filled with cubic centimetre cubes. To count the cubes, it is sufficient to fill the box partially to find the counting strategy and to understand the formula. Number of cubes in the box = number of cubes in a layer - number of layers. The next step is to move on to the dimensional number formula: Measure of the volume = measure of the length - measure of the width - measure of the height (if length, width, and height have the same unit of measurement)	 Every layer has 5x3 = 15 unit cubes. 4 layers fit into the box to fill it up. $V = 5 \cdot 3 \cdot 4\ cm^3 = 60\ cm^3$ 09_Koerpergeometrie.pdf (hu-berlin.de) (Filler)  Source: Heide Kühne

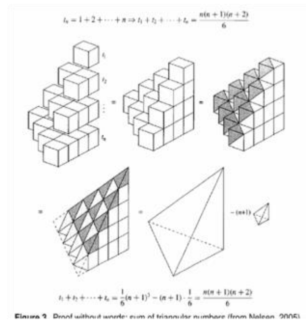
7	Parallel	In three-dimensional Euclidean space, two lines are parallel if they lie in the same plane and coincide and do not intersect.	The opposite sides of a cuboid are parallel to each other.	
		A straight line is parallel to a plane if it lies entirely in this plane or does not intersect it.	The edges of a cuboid which are not in the basic square are parallel to this square.	
		Two planes are parallel if they coincide or do not intersect. This is referred to as parallel planes.	The opposite surfaces of a cuboid are parallel.	

8	Distance	A numerical or sometimes qualitative measurement of how far apart geometrical objects or points are.	For example the distance between cutting planes in our example is 1 cm.	
9	Length unit	A **unit of length** refers to any arbitrarily chosen and accepted reference standard for measurement of length.	In our examples we measure length with cm.	 Source: Heide Kühne

10	Volume unit	A unit of volume is equal to the volume occupied by a unit cube with a side length of one. Since the volume occupies three dimensions, if L is chosen as a unit of length, the corresponding unit of volume is L3	Since we have chosen cm to measure length, we can use cm3 to measure volume of a cube in our example. The volume of a cube is 64 cm3	 Source: Heide Kühne

Didactical Terms (Geometry)

	Terminology	Definition	Examples	Illustration
1	Experimenting with geometrical models	Students can produce, analyze, describe or identify models of geometrical solids by using all of their senses.	In our example, we suggest giving learners the possibility to build cube models from clay or soap and to divide them into unit cubes to estimate the volume.	
2	Sketching and drawing	Students can use linguistic strategies of signed languages to produce 3D-sketches, use GeoGebra or video to create dynamic visualization of solids or draw 2D-sketches using paper and pencil.	In our example we show concrete GeoGebra-images and animations of cube divisions.	
3	Measuring	To measure the volume of solid transparent models, students can first measure their extensions by determining height, length, and width. The transparent solids can also be filled with water, sand or different unit cubes to estimate the volume.	In our example, an empty and transparent cube with the same measures as in the example can be filled with 64 unit cubes which have a volume measurement of 1 cm3.	 09_Koerpergeometrie.pdf (hu-berlin.de) (Filler)

4	Calculating	To calculate the measurements of solids, a formula which was derived from geometrical proofs can be used.	In our example, learners can use the appropriate formula to calculate the volume of a cube or cuboid.	**Formel:** $V = a \cdot b \cdot c$ Source: Heide Kühne $V = 5{\cdot}3{\cdot}4\ cm^3 = 60\ cm^3$
5	Geometrical Reasoning	After the students are done experimenting, measuring, calculating and documenting their findings, they can try to reflect on the geometrical arguments they used and try to generalize their findings. To prove mathematical statements, it could be helpful to use the visuals suggested by Nelson (2005) in his book "Proofs Without Words." However we do recommend discussing and documenting visual proofs in signed languages and not to rely on the visuals alone. In our paper we give an explanation on how concrete and visual images or pictures can be embedded in the language and, by these means, allow generalizations of mathematical propositions giving not only concrete pictorial images, but also arguments to communicate and to think about geometrical proofs.	In our example, students can try to find the formula which best helps them to estimate the volume of specific pyramid and prism types by dividing the cubes held therewithin. From these special cases, they can move on to general methods for the estimation of the volume of prisms and pyramids.	 Source: Nelson (2005) *Proofs Without Words*

References

Affolter W., Amstad, H., Beerli, G., Doebeli, M., Hurschler H., Jaggi, B., Jundt, W., Krummenacher, R., Nydegger, A., Wälti, B. & Wieland, G. (2011). *Das Mathematikbuch 7*. Ernst Klett Verlag.

Angeloni, F. (2023). Gebärden über Variablen unter dem Gegenstandsaspekt. *Beiträge zum Mathematikunterricht 2022* (pp. 529–532). WTM Verlag. https://doi.org/10.17877/DE290R-23545.

Angeloni, F. (submitted). Die Ikonizität der Gebärden über Variablen unter dem Einsetzungsaspekt. *Beiträge zum Mathematikunterricht 2024*.

Angeloni, F., Wille, A. M., & Hausch, C. (2022). Signing about elementary algebra in Austrian Sign Language: What signs of the notion of variable can represent. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti. (Eds.), *Proceedings of the Twelfth Congress of European Research Society in Mathematics Teaching (CERME12)* (pp.4218–4225). Free University of Bozen-Bolzano and ERME. https://hal.science/hal-03765017 .

Angeloni, F., & Wille, A. M. (2022). Bimodal-bilinguale Lernumgebungen: der Satz des Pythagoras in Österreichischer Gebärdensprache. *Das Zeichen, 36*(118), 134-147. Retrieved from https://www.das-zeichen.online/hefte/dz-118/bimodal-bilinguale-lernumgebungen-der-satz-des-pythagoras-in-oesterreichischer-gebaerdensprache/.

Angeloni, F., Wille, A. M., & Hausch, C. (2023). Representation of numbers and variables in Austrian Sign Language. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 4385–4392). Alfréd Rényi Institute of Mathematics and ERME.

Aspinwall, L., Shaw, K.L., & Presmeg, N.C. (1997). Uncontrollable mental imagery: graphical connections between a function and its derivative. Educational Studies in Mathematics, 33, 301-317, https://doi.org/10.1023/A:1002976729261.

Barth, I., Illmer, B., Robert Jasko, R., Löffler, J., & Uta, M. (2022). Entwicklung eines MINT-Fachgebärdenlexikons: Von der Idee bis zur Umsetzung des „Sign2MINT“-Projekts. *Das Zeichen, 36*(119), 150–176.

Becker, C. (2019). Inklusive Sprachbildung. Impulse aus der Gebärdensprach- und Audiopädagogik. In L. Rödel, & T. Simon, *Inklusive Sprach(en)bildung. Ein interdisziplinärer Blick auf das Verhältnis von Inklusion und Sprachbildung*, 72- 86. Forschung Klinkhardt. Retrieved from https://elibrary.utb.de/doi/pdf/10.35468/9783781557512.

Becker, N. (2006). *Die neurowissenschaftliche Herausforderung der Pädagogik.* Klinkhardt Forschung. Retrieved from https://www.pedocs.de/volltexte/2012/5580/pdf/Becker_2006_Neurowissenschaftliche_Herausforderung_D_A.pdf.

Blatto-Vallee, G., Kelly, R. R., Gaustad, M. G., Porter, J., & Fonzi, J. (2007). Visual spatial representation in mathematical problem solving by deaf and hearing students. Journal of Deaf Studies and Deaf Education, 12(4), 432–448, https://doi.org/10.1093/deafed/enm022

Bogdanova, T.G. (2021).Žestovaja reč' i psichičeskoe razvitie gluchich detej: sovremennoe sostojanie voprosa (obzor zarubežnych issledovanij), Kliničeskaja i special'naja psichologija, 10 (2), 3–22, https://doi.org/10.17759/cpse.2021100202.

Bogdanova, T. (2021). Sign Language and Psychological Development of Deaf Children: State-of-the-Art (Foreign Studies Review). *Clinical psychology and special education, 10*(2), 3-22. Retrieved from https://orcid.org/0000-0002-5886-6494.

Brennan, M. (2000) Fair assessment for deaf candidates: A report to the Scottish Qualifications Authority. University of Edinburgh. https://www.sqa.org.uk/files_ccc/FairAssessment_MaryBrennan.pdf

Brien. D. (ed) (1993) The Dictionary of British Sign Language, Faber & Faber, London.

Cameron, A., Quinn, G. and O'Neill, R (2012) Development of Physics and Engineering Signs in British Sign Language. Final Report to the STEM Disability Committee and the Royal Academy of Engineering. SSC.

Cameron, A. (2015) The development of astronomy signs and analysis of impact on deaf and hearing communities. Conference paper: Projeto Surdos: Simposio Caminhos da Inclusao at UFRJ, Rio de Janeiro. https://www.researchgate.net/publication/281593745_The_development_of_astronomy_signs_and_analysis_of_impact_on_deaf_and_hearing_communities

Cameron, A., O'Neill, R., & Quinn, G. (2017). Deaf students using sign language in mainstream science classrooms. In Oliveira, A. & Weinburgh, M. (Eds.), Science Teacher Preparation in Content-Based Second Language Acquisition (pp. 341-360). Springer.. https://doi.org/10.1007/978-3-319-43516-9\

Cameron, A., O'Neill, R. and Quinn, G. (2019) 'Deaf Scientists create new technical terminology in British Sign Language', Physiology News, v115, p26.

Cameron, A. (2024). Signing to Know research (2021-2026) – classroom observations of dialogue.

Campbell, K.J., Collis, K.F., & Watson, J. (1995). Visual processing during mathematical problem solving. Educational Studies in Mathematics, 28, 177-194, https://doi.org/10.1007/BF01295792.

Chen, L., & Wang, Y. (2020). The contribution of general cognitive abilities and specific numerical abilities to mathematics achievement in students who are deaf or hard-of-hearing. Journal of Developmental and Physical Disabilities, 33, 771 – 787, https://doi.org/10.1007/s10882-020-09772-8.

Chen, L. (2022). Spatial ability and mathematics achievement in deaf children: the mediating role of processing Speed and Intelligence. Journal of Developmental and Physical Disabilities 34, 399–415, https://doi.org/10.1007/s10882-021-09805-w.

Clark, K., Sheikh, A., Swartzenberg, J., Gleason, A., Cummings, C., Dominguez, J., Mailhot, M., & Collison, C. G. (2021). Sign Language Incorporation in Chemistry Education (SLICE): Building a Lexicon to Support the Understanding of Organic Chemistry. Journal of chemical education, 99(1), 122–128. https://doi.org/10.1021/acs.jchemed.0c01368

Cohen, S. (2024). Using a Language Community to Unlock the Abstractness of Signed Language. Journal of Deaf Studies and Deaf Education, 29(2), 282-283.

D'jačkov, A. (1961). Cistemy obučenija gluchich detej. Akademija pedagogičeskich nauk RSFCR.

De Jong, O. and Taber, K. S. Teaching and Learning the Many Faces of Chemistry. In Handbook of research in science education; Abell, S. K., Lederman, N. G., Eds.; Routledge: New York, 2007; pp 631– 652.

Devlin, K. J. (1998). *Muster der Mathematik: Ordnungsgesetze des Geistes und der Natur.* Spektrum.

Di Luca, S., & Pesenti, M. (2011). Finger numeral representations: More than just another symbolic code. *Frontiers in psychology, 2*, 28-30. Retrieved from https://doi.org/10.3389/fpsyg.2011.00272.

Driver, R., Asoko, H., Leach, J., Scott, P., & Mortimer, E. (1994). Constructing Scientific Knowledge in the Classroom. Educational Researcher, 23(7), 5-12.

Driver, R., Squires, A., Rushworth, P., & Wood-Robinson, V. (2014). Making Sense of Secondary Science: Research into children's ideas (2nd ed.). Routledge. https://doi.org/10.4324/9781315747415

Edwards, A., Edwards, L. & Langdon, D. (2013). The mathematical abilities of children with cochlear implants. *Child neuropsychology: a journal on normal and abnormal development in childhood and adolescence*, *19*(2), 127–142. https://doi.org/10.1080/09297049.2011.639958.

Emmorey K. (2023). Ten things you should know about sign languages. Current directions in psychological science, 32(5), 387–394, https://doi.org/10.1177/09637214231173071

Enderle, P., Cohen, S., & Scott, J. (2020). Communicating about science and engineering practices and the nature of science: An exploration of American Sign Language resources. Journal of Research in Science Teaching, 57(6), 968-995.

Fleri, V. I. (1835). *Deaf-mutes considered in relation to their condition and to the modes of education that are inherent in their nature.* Plushar.

Flores, A. and Rumjanek, V. (2015) Teaching Science to Elementary School Deaf Children in Brazil. Creative Education, 6, 2127-2135. doi: 10.4236/ce.2015.620216.

Frick, A. (2019) Spatial transformation abilities and their relation to later mathematics performance. Psychological Research, 83, 1465–1484, https://doi.org/10.1007/s00426-018-1008-5.

Gates, P. (2017). The Importance of Diagrams, Graphics and Other Visual Representations in STEM Teaching. In STEM Education in the Junior Secondary (pp. 169–196). Springer Singapore Pte. Limited.

Groninger, H. Sieprath, H. (2019). Visuelles Begreifen mit der Signcreative Spiel- und Lernplattform. In S. Hornäk et. al. (Eds.). In der Praxis: Inklusive Möglichkeiten künstlerischen und kunstpädagogischen Handelns (pp. 71-80). Kopaed.

Grote, K., & Kramer, F. (2009). Haben Gehörlose beim Rechnen mehr Schwierigkeiten als Hörende? *Das Zeichen, 22*(82), 276-283.

Grote, K., & Linz, E. (2003). The influence of sign language iconicity on semantic conceptualization. In W. G. Olga Fischer, *From sign to signing*, 23–40. John Benjamins. Retrieved from https://doi.org/10.1075/ill.3.

Grote, K., Sieprath, H., & Staudt, B. (2018). Deaf Didaktik? Weshalb wir eine spezielle Didaktik für den Unterricht in Gebärdensprache benötigen. In *DAS ZEICHEN. Zeitschrift für Sprache und Kultur Gehörloser. Vol. 110* (pp. 426–437).

Grote, K., Sieprath, H. & Staudt, B. (2018). Deaf Didaktik? Weshalb wir eine spezielle Didaktik für den Unterricht in Gebärdensprache benötigen. *Das Zeichen: Zeitschrift für Sprache und Kultur Gehörloser, 32*(110), 2-13.

Hansen, E. G., Loew, R. C., Laitusis, C. C., Kushalnagar, P., Pagliaro, C. M., & Kurz, C. (2018). Usability of American Sign Language Videos for Presenting Mathematics Assessment Content. *The journal of Deaf Studies and Deaf Education*, *23*(3), 284–294. Retrieved from https://doi.org/10.1093/deafed/eny008.

Hänel-Faulhaber, B., Schäfer, K., Werner, V. (2023). *Mathematiklernen im Förderschwerpunkt Hören und Kommunikation. Leitlinien guter Unterricht.* Arbeitskreis Unterricht, Berufsverband Deutscher Hörgeschädigtenpädagogen e. V. https://bdh-guter-unterricht.de/fachdidaktiken/mathematik.

Henner, J., Pagliaro, C., Sullivan, S., & Hoffmeister, R. (2021). Counting Differently: Assessing Mathematics Achievement of Signing Deaf and Hard of Hearing Children Through a Unique Lens. *American annals of the Deaf, 166*(3), 318-341. Retrieved from https://doi.org/10.1353/aad.2021.0023.

Hickman, J. (2013). Using British Sign Language (BSL) in science education. Guest Royal Society of Biology Blog https://blog.rsb.org.uk/using-british-sign-language-bsl-in-science-education/

Higgins, J. A., Famularo, L., Cawthon, S. W., Kurz, C. A., Reis, J. E., & Moers, L. M. (2016). Development of American Sign Language Guidelines for K-12 Academic Assessments. *The journal of Deaf Studies and Deaf Education, 21*(4), 383-393. Retrieved from https://doi.org/10.1093/deafed/enw051.

Höst G., Schönborn K.J., and Tibell L. (2022) Visual images of the biological microcosmos: Viewers' perception of realism, preference, and desire to explore. Front. Educ. 7:933087. doi 10.3389/feduc.2022.933087

Jaškova, N. V. (1998) Nagljadnoe myšlenie gluchich detej. Pedagogika.

Jäger, J., Schupp, H. (1983). Curriculum Stochastik in der Hauptschule. Schöningh.

Johnson, M. (1987). The body in the mind: The bodily basis of meaning, imagination, and reason. University of Chicago Press.

Johnstone A. H., (1991), Why is science difficult to learn? Things are seldom what they seem, J. Comput. Assist. Lear., 7, 75–83.

Joice, W. and Tetlow, A. (2021) Jisc: Disability STEM data for students and academic staff in higher education 2007/08 – 2018/19 – research conducted on behalf of the Royal Society. p9. https://royalsociety.org/-/media/policy/topics/diversity-in-science/210118-disability-stem-data-for-students-and-staff-in-higher-education.pdf https://royalsociety.org/about-us/who-we-are/diversity-inclusion/disability-reports/

Jones, L. (2014). Developing Deaf Children's Conceptual Understanding and Scientific Argumentation Skills: A Literature Review. Deafness & Education International, 16(3), 146–160. https://doi.org/10.1179/1557069X13Y.0000000032

Känguru der Mathematik. (2022). Adventskalender in Gebärdensprache. https://www.mathe-kaenguru.de/advent/gebaerden/index.html.

Khokhlova, A. (2013). Role of sign language in intellectual and social development of deaf children: Review of foreign publications. *Journal of modern foreign psychology, 2*(4), 59–68. Retrieved from https://psyjournals.ru/en/journals/jmfp/archive/2013_n4/65488.

Kiernan, N., Manches, A., & Seery, M. K. (2021). The role of visuospatial thinking in students' predictions of molecular geometry. Chemistry Education Research and Practice, v22, p626-639. https://doi.org/10.1039/D0RP00354A

Kiernan, N., Manches, A., & Seery, M. (2024) Resources for reasoning of chemistry concepts: multimodal molecular geometry. Chem. Educ. Res. Pract., 2024, 25, 524

Kolmogorov, A. N. (2001). O razwitii matematičeskich sposobnostej. Pis'mo V. A. Kruteckomu, Voprosy psichologii, 3, 103-106, https://kolmogorov.info/o_razvitii_matemat_sposobnostey.html

Krajewski, K. (2018). *MBK 0. Test mathematischer Basiskompetenzen im Kindergartenalter.* Hogrefe.

Korvorst, M., Nuerk, H.-C., & Willmes, K. (2007). The Hands Have It: Number Representations in Adult Deaf Signers. *The journal of Deaf Studies and Deaf Education, 12*(3), 362-372. Retrieved from https://doi.org/10.1093/deafed/enm002.

Krause, C. M. (2016). DeafMath – Ein Projekt zum Einfluss der Gebärdensprache auf Mathematikverständnis. *Beiträge zum Mathematikunterricht 2016. Vorträge auf der 50. Tagung für Didaktik der Mathematik vom 07.03.2016 bis 11.03.2016 in Heidelberg* (pp. 577–580). WTM Münster.

Krause, C. M. (2017). DeafMath: Exploring the influence of sign language on mathematical conceptualization. In Dooley, T., & Gueudet, G. (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10, February 1-5, 2017)* (pp. 1316–1323). Dublin, Ireland: DCU Institute of Education and ERME.

Krutetskii, V. A. (1976). The psychology of mathematical abilities in schoolchildren. University of Chicago Press.

Kultusministerkonferenz. (2003). Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss. Beschluss vom 4.12.2003. https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2003/2003_12_04-Bildungsstandards-Mathe-Mittleren-SA.pdf.

Kultusministerkonferenz. (2004). Bildungsstandards im Fach Mathematik für den Primarbereich. Beschluss vom 15.10.2004, i.d.F. vom 23.06.2022. https://www.kmk.org/fileadmin/Dateien/veroeffentlichungen_beschluesse/2022/2022_06_23-Bista-Primarbereich-Mathe.pdf.

Kurz, C., & Pagliaro, C. M. (2019). Using L1 sign language to teach mathematics. In The Routledge handbook of sign language pedagogy (pp. 85-99). Routledge.

Kusters, A., & Hou, L. (2020). Linguistic Ethnography and Sign Language Studies. Sign Language Studies 20(4), 561-571. https://doi.org/10.1353/sls.2020.0018.

Kutscher, S. (2010). Ikonizität und Indexikalität im gebärdensprachlichen Lexikon – Zur Typologie sprachlicher Zeichen. In *Zeitschrift für Sprachwissenschaft.* Vol. 29(1) (pp. 79–109). Walter de Gruyter GmbH. https://doi.org/10.1515/zfsw.2010.00.

Lang, H. G., Hupper, M. L., Monte, D. A., Brown, S. W., Babb, I., & Scheifele, P. M. (2007). A study of technical signs in science: implications for lexical database development. Journal of deaf studies and deaf education, 12(1), 65–79. https://doi.org/10.1093/deafed/enl018

Langdon, C., C Kurz, C., & Coppola, M. (2023). The Importance of Early Number Concepts for Learning Mathematics in Deaf and Hard of Hearing Children. *Perspectives on early childhood psychology and education, 5*(2). Retrieved from https://digitalcommons.pace.edu/cgi/viewcontent.cgi?article=1061&context=perspectives.

Leikin, R. (2021). When practice needs more research: the nature and nurture of mathematical giftedness. Zdm, 53, 1579 – 1589, https://doi.org/10.1007/s11858-021-01276-9.

Leybaert, J., & Van Cutsem, M.-N. V. (2002). Counting in sign language. *Journal of experimental child psychology, 81*(4), 482–501. Retrieved from https://doi.org/10.1006/jecp.2002.2660.

Loots, G., Devisé, I., & Jacquet, W. (2005). The impact of visual communication on the intersubjective development of early parent-child interaction with 18- to 24-month-old deaf toddlers. *The journal of Deaf studies and Deaf education, 10*(4), 357–375. Retrieved from https://doi.org/10.1093/deafed/eni036.

Lindahl, C. (2015). Tecken av betydelseâ€¨: En studie av dialog i ett multimodalt, Teckenspråkigt tvåspråkigt NO-klassrum. (PhD dissertation, Institutionen för matematikämnets och naturvetenskapsämnenas didaktik, Stockholms universitet). Retrieved from https://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-119363

Lindahl, C. (2021). 6 Sign Bilingualism as Semiotic Resource in Science Education: What Does It Mean?. In K. Snoddon & J. Weber (Ed.), Critical Perspectives on Plurilingualism in Deaf Education (pp. 129-148). Bristol, Blue Ridge Summit: Multilingual Matters. https://doi.org/10.21832/9781800410756-009

Lualdi, C.P., Spiecker, B., Wooten, A.K. et al. Advancing scientific discourse in American Sign Language. Nat Rev Mater 8, 645–650 (2023). https://doi.org/10.1038/s41578-023-00575-9

Lualdi, C. P., Spiecker, B., & Clark, A. K. (2023). Advancing scientific discourse in American Sign Language. *Nature Reviews Materials, 8*(10), 645-650. Retrieved from https://www.nature.com/articles/s41578-023-00575-9.

Malle, G. (1993). *Didaktische Probleme der elementaren Algebra*. Springer.

Marschark, M. & Knoors, H. (2012). Sprache, Kognition und Lernen – Herausforderungen an die Inklusion gehörloser und schwerhöriger Kinder. In M. Hintermair (Ed.), *Inklusion und Hörschädigung. Diskurse über das Dazugehören und Ausgeschlossensein im Kontext besonderer Wahrnehmungsbedingungen* (pp. 129-176). Median.

Marschark, M., Paivio, A., Spencer, L. J., Durkin, A., Borgna, G., Convertino, C., & Machmer, E. (2017). Don't Assume Deaf Students are Visual Learners. *Journal of developmental and physical disabilities, 29*(1), 153-171. Retrieved from https://doi.org/10.1007/s10882-016-9494-0.

Marschark, M., & Knoors, H. (2012). Sprache, Kognition und Lernen – Herausforderungen an die Inklusion gehörloser und schwerhöriger Kinder. In M. Hintermair, *Diskurs über das Dazugehören und Ausgeschlossensein im Kontext besonderer Wahrnehmungsbedigungen*, 129–176. Median.

Marschark, M., Spencer, L. J., Durkin, A., Borgna, G., Convertino, C., Machmer, E., Kronenberger, W. G., & Trani, A. (2015). Understanding language, hearing status, and visual-spatial skills. Journal of Deaf Studies and Deaf Education, 20(4), 310–330, https://doi.org/10.1093/deafed/env025

Masataka, N., Ohnishi, T., Imabayashi, E., Hirakata, M., & Matsuda, H. (2006). Neural Correlates for Numerical Processing in the Manual Mode. *The journal of Deaf Studies and Deaf Education,* *11*(2), 144–152. Retrieved from https://doi.org/10.1093/deafed/enj017.

Masataka, N. (2006). Differences in Arithmetic Subtraction of Nonsymbolic Numerosities by Deaf and Hearing Adults. *The journal of Deaf Studies and Deaf Education, 11*(2), 139-143. Retrieved from https://doi.org/10.1093/deafed/enj016.

Meara, R., Cameron, A., Quinn, G., and O'Neill, R. (2016) Development of Geography and Geology Terminology in British Sign Language. Geophysical Research Abstracts, Vol. 18, EGU2016-12390, 2016. EGU General Assembly 2016 https://meetingorganizer.copernicus.org/EGU2016/EGU2016-12390.pdf

Mercer, N., & Littleton, K. (2007). Dialogue and the development of children's thinking : a sociocultural approach. Routledge, London & New York.

Morford, J. (1996). Insights to Language from the Study of Gesture: a Review of Research on the Gestural Communication of Non-signing Deaf People. *Language & communication, 16*(2), 165—178. Retrieved from https://doi.org/10.1016/0271-5309(96)00008-0.

NASA Science (n.d.)

Mercury Facts: https://science.nasa.gov/mercury/

Venus Facts: https://science.nasa.gov/venus/

Earth Facts: https://science.nasa.gov/earth/facts/

Mars Facts: https://science.nasa.gov/mars/

Jupiter Facts: https://science.nasa.gov/jupiter/

Saturn Facts: https://science.nasa.gov/saturn/

Uranus Facts: https://science.nasa.gov/uranus/facts/

Neptune Facts: https://science.nasa.gov/neptune/facts/

Neville, H. J., Bavelier, D., Corina, D., Rauschecker, J., Karni, A., Lalwani, A., Turner, R. (1998). Cerebral organization for language in deaf and hearing subjects: biological constraints and effects of experience. *Proceedings of the National Academy of Sciences of the United States of America, 95*(3), 922–929. Retrieved from https://doi.org/10.1073/pnas.95.3.922.

Nineteenth Annual Report of the Columbia Institution for the Deaf and Dumb. (1876). *Gallaudet University Annual Report from 1858 to 1967*. Government Printing Office. Retrieved from https://archive.org/details/ANNUALREPORT-Gallaudet-1876/mode/2up.

Niss, M. (2019). The very multi-faceted nature of mathematics education research. For the Learning of Mathematics, 39(2), 2-7, https://www.jstor.org/stable/26757463.

Nordheimer, S., Marlow, A., Scholtz, J. (2024, 20.-23. February). Fostering mathematical creativity and talents with mathematical problems and competitions in German Sign Language. [Conference contribution]. The 13th International Group for Mathematical Creativity and Giftedness (IMCGC), Bloemfontein, South Africa.

Nunes, T. (2020). Deaf Children, Special Needs, and Mathematics Learning. In S. Lerman, *Encyclopedia of Mathematics Education* (pp. 181-183). Springer, Cham. Retrieved from https://doi.org/10.1007/978-3-030-15789-0_42.

Nunes, T. (2004). *Teaching mathematics to deaf children.* Whurr.

Nunes, T., & Moreno, C. (2002). An intervention program for promoting deaf pupils' achievement in mathematics. *The journal of Deaf studies and Deaf Education, 7*(2), 120-133. Retrieved from https://doi.org/10.1093/deafed/7.2.120.

Nunes, T., & Moreno, C. (1998). The Signed Algorithm and Its Bugs. *Educational studies in mathematics, 35*(1), 85-92. Retrieved from https://doi.org/10.1023/A:1003061009907.

O'Neill, R., Quinn, G., & Cameron, A. (2019). Learning the lingo: How deaf scientists create new technical terminology in British Sign Language. Physiology News, (115), 26. https://www.physoc.org/magazine-articles/learning-the-lingo-how-deaf-scientists-create-new-technical-terminology-in-british-sign-language/

O'Neill, R., Cameron, A., Burns, E., & Quinn, G. (2020). Exploring alternative assessments for signing deaf candidates. Psychology in the Schools, 57(3), 344-361.

Pabis, S., & Catalano, J. (2023). Explicit and Contextualized Math Vocabulary Instruction With Deaf and Hard-of-Hearing Students,. *The journal of Deaf Studies and Deaf Education, 28*(4), 424–425. Retrieved from https://doi.org/10.1093/deafed.

Pagliaro, C. M., & Kritzer, K. L. (2013). The Math Gap: a description of the mathematics performance of preschool-aged deaf/hard-of-hearing children. Journal of Deaf Studies and Deaf Education, 18(2), 139–160, https://doi.org/10.1093/deafed/ens070.

Papaspyrou, C., Meyenn, A. v., Matthaei, M., & Herrmann, B. (2008). *Grammatik der Deutschen Gebärdensprache aus der Sicht gehörloser Fachleute.* Signum.

Parasnis, I., Samar, V.J., Bettger, J., & Sathe, K. (1996). Does deafness lead to enhancement of visual spatial cognition in children? Negative evidence from deaf nonsigners. Journal of deaf studies and deaf education, 1(2), 145-52, https://doi.org/10.1093/oxfordjournals.deafed.a014288.

Pitta-Pantazi, D., & Christou, C. (2010). Spatial versus object visualisation: The case of mathematical understanding in three-dimensional arrays of cubes and nets. International Journal of Educational Research, 49, 102-114, https://doi.org/10.1016/j.ijer.2010.10.001

Polya, G. (1969). Mathematik und Plausibles Schließen. Wissenschaft und Kultur.

Presmeg, N.C. (1986). Visualisation and mathematical giftedness. Educational Studies in Mathematics, 17, 297-311, https://doi.org/10.1007/BF00305075.

Rainò, P., & Halkosaari, O. A. (2018). The deaf way of interpreting mathematical concepts. In C. Stone (Ed), *Deaf interpreting in Europe. Exploring best practice in the field*, 10-20. Danish Deaf Association. Retrieved from https://ddl.dk/wp-content/uploads/DEAF-INTERPRETING-IN-EUROPE-2017.pdf.

Rathmann, C. (2025). An example of a task from stochastics with German Sign language.

Rathmann, C. (2022, 10. February). Creation of technical signs in German Sign Language. Iconic and linguistic strategies and application. Workshop Signs Neologisms. Athens, Greece.

Rathmann, C., Mann, W. and Morgan, G. (2007). Narrative structure and narrative development in deaf children. *Deafness Educ. Int., 9*, 187-196.

Raven, S., & Whitman, G. M. (2019). Science in Silence: How Educators of the Deaf and Hard-of-Hearing Teach Science. Research in Science Education (Australasian Science Education Research Association), 49(4), 1001–1012. https://doi.org/10.1007/s11165-019-9847-7

Reid, N. (2021). The Johnstone triangle : the key to understanding chemistry / Norman Reid. Royal Society of Chemistry.

Roth, J. (2013). Vernetzen als durchgängiges Prinzip – Das Mathematik-Labor „Mathe ist mehr". In A. S. Steinweg (Ed.), *Mathematik vernetzt. Band 3 der Reihe „Mathematikdidaktik Grundschule"* (pp. 65-80), University of Bamberg Press.

Rozanova, T. V (1991). Razwitie sposobnostej gluchich detej. Pedagogika.

Rosanova, T. (1991). *The development of abilities in deaf children.* Pedagogika.

Rozanova, T. V. (1978). Razwitie pamjati i myšlenija gluchich detej. Pedagogika.

Rosanova, T. (1978). *The development of memory and thinking of deaf children.* Pedagogika.

Rosanova, T. (1971). Memory. In I. Solovjev, Z. Shif, T. Rosanova, & N. Yashkova, *The psychology of deaf children*, 87-124. Enlightenment.

Rozanova, T. V. (1966). Psichologija rešenija zadač gluchimi škol'nikami. Prosveščenie.

Ruf, U., & Gallin, P. (1999). *Ich mache das so! Wie machst du es? Das machen wir ab. Sprache und Mathematik, 4.-5. Schuljahr bzw. 5.-6. Schuljahr.* Lehrermittelverlag.

Santos, S., Brownell, H., Coppola, M., & Cordes, A. S. (2023). Language experience matters for the emergence of early numerical concepts. *NPJ science of learning, 8*(1). Retrieved from https://doi.org/10.1038/s41539-023-00202-w.

Santos, S., & Cordes, S. (2022). Math abilities in deaf and hard of hearing children: The role of language in developing number concepts. *Psychological Review*, *129*(1), 199-211. Retrieved from https://doi.org/10.1037/rev0000303.

Scott, J., Henner, J., & Skyer, M. (2023). Six Arguments for Vygotskian Pragmatism in Deaf Education: Multimodal Multilingualism as Applied Harm Reduction. *American annals of the Deaf, 168*(1), 56-79.

Schäfer, Karolin & Gohmann, Laura & Westerhoff, Helena & Schindler, Maike. (2022). Gebärdensprachvideos als Hilfe beim Bearbeiten mathematischer Textaufgaben bei gehörlosen Schüler:innen: Eine schulische Intervention mit Tablet-Computern. 118, 1-18.

Secora, K., & Emmorey, K. (2020). Visual-Spatial Perspective-Taking in Spatial Scenes and in American Sign Language. Journal of deaf studies and deaf education, 25(4), 447–456, https://doi.org/10.1093/deafed/enaa006.

Sign Language Dictionary (2018). SpreadTheSign. https://www.spreadthesign.com.

Sill, H.-D. & Kurtzmann, G. (2019). *Didaktik der Stochastik in der Primarstufe.* Springer Spektrum.

Skyer, M.E. (2023). Multimodal transduction and translanguaging in deaf pedagogy. Languages, 8(2), 27-167, https://doi.org/10.3390/languages8020127.

Soeharto, S., & Csapó, B. (2021). Evaluating item difficulty patterns for assessing student misconceptions in science across physics, chemistry, and biology concepts. Heliyon, 7(11).

STEM in Sign Language. (2023). Mathe-Adventskalender in DGS. https://stemsil.eu/mathe-adventskalender/?lang=de.

Sture, T. K. (1984) Osobennosti razwitija myšlenija gluchich učaščichsja pri rešenii zabač po fizike. Akademija pedagogičeskich nauk RSFSR.

Suchova, V. B. (1966). Obučenie nagljadnoj geometrii v škole dlja gluchich. Akademija pedagogičeskich nauk RSFSR.

Suchova, V. (1966). *Teaching visual geometry in a school for the deaf.* Academy of Pedagogical Sciences of the RSFSR.

Suchova, V. (2002). Teaching Mathematics in Preparatory-IV Grades of Schools for Deaf and Hard of Hearing Children. Textbook for universities / V.B. Sukhova. - Moscow : Academy; 184.

Szűcs, K. (2019, 6.–10. February). Do hearing-impaired students learn mathematics in a different way than their hearing peers? – A review. *Eleventh Congress of the European Society for Research in Mathematics Education.* Utrecht, Netherlands.

Tabak, J. (2014). What Is Higher Mathematics? Why Is It So Hard to Interpret? What Can Be Done? *Journal for interpretation, 23*(1), 1-18. Retrieved from https://digitalcommons.unf.edu/joi/vol23/iss1/5.

Tabak, J. (2016). On the Expression of Higher Mathematics in American Sign Language. *Journal of interpretation, 25*(1), 1-19. Retrieved from https://digitalcommons.unf.edu/joi/vol25/iss1/10.

Taber, K. S. (2013). Revisiting the chemistry triplet: drawing upon the nature of chemical knowledge and the psychology of learning to inform chemistry education. Chemistry Education Research and Practice, 14(2), 156-168.

Taber, K. (2012). Teaching secondary chemistry. London: Hodder Education.

Thom, J.S., & Hallenbeck, T. (2021). Spatial reasoning in mathematics: A cross-field perspective on deaf and general education research. Deafness & Education International, 24, 127–159, https://doi.org/10.1080/14643154.2020.1857539.

Quinn, G., Cameron, A., and O'Neill, R. (2021). Signing the times: Enforced oralism in deaf education has left BSL lacking signs for specialist concepts. NEWSLI Magazine (UK), pp27-31.

Vernon, M., & Wallrabenstein, J. M. (1984). The Diagnosis of Deafness in a Child. *Journal of communicational disorders, 17*(1), 1-8.

Vernon, M. (2005). Fifty Years of Research on the Intelligence of Deaf and Hard-of-Hearing Children: A Review of Literature and Discussion of Implications. *The journal of Deaf Studies and Deaf Education, 10*(3), 225—231.

Villwock, A., Wilkinson, E., Piñar, P., & Morford, J. P. (2021). Language development in deaf bilinguals: Deaf middle school students co-activate written English and American Sign Language during lexical processing. *Cognition,* 211. Retrieved from https://doi.org/10.1016/j.cognition.2021.104642.

Walker, K., Carrigan, E., & Coppola, M. (2024). Early access to language supports number mapping skills in deaf children. *The journal of Deaf Studies and Deaf Education, 29*(1), 1-18. Retrieved from https://doi.org/10.1093/deafed/enad045.

Wauters, L., Pagliaro, C., Kritzer, K. L., & Dirks, E. (2023). Early mathematical performance of deaf and hard of hearing toddlers in family-centred early intervention programmes. *Deafness & Education International, 26*(2), 190–207. Retrieved from https://doi.org/10.1080/14643154.2023.2201028.

Weber, D., Beumann, S., & Benölkken, R. (2023, 10.-14. July). Teachers' view of twice-exceptional students – outline of the challenges in recognizing mathematical giftedness and supporting needs of hearing impairment. Conference: Proceedings of the 13th Congress of the European Society for Research in Mathematics Education (CERME), Budapest, Hungary.

Weigand, H.-G., Filler, A., Hölzl, R., Kuntze, S., Ludwig, M., Roth, J., Schmidt-Thieme, B., & Wittmann, G. (2018). Didaktik der Geometrie für die Sekundarstufe I. Springer Spektrum.

Werner, V., & Hänel-Faulhaber, B. (2024, 7.-14. July). Do Numbers in German Sign Language Support the Development of Counting Skills? *15th International Congress on Mathematical Education.* Sydney, Australien.

Werner, V., & Hänel-Faulhaber, B. (2024). Numerische Kompetenzen tauber Schulanfänger_innen in Deutscher Gebärdensprache (DGS). *Lernen und Lernstörungen, 13*(3), 123-135. Retrieved from https://doi.org/10.1024/2235-0977/a000449.

Werner, V., & Hänel-Faulhaber, B. (2022, 29. August - 02. September). Löseverhalten tauber und hörender Kinder beim Reparieren von Reihen- und Kreismusterfolgeaufgaben – erste Ergebnisse. *56. Jahrestagung der Gesellschaft für Didaktik der Mathematik (GDM).* Frankfurt am Main, Germany.

Werner, V. (2010). Zum numerischen Zahlenverständnis von gehörlosen Grundschülern (Teil II). *Das Zeichen, 24*(85), 276–289. Retrieved from https://www.das-zeichen.online/hefte/dz-085/zum-numerischen-zahlenverstaendnis-von-gehoerlosen-grundschuelern/.

Werner., V., Masius, M., Ricken, G., & Hänel-Faulhaber, B. (2019). Mathematische Konzepte bei gehörlosen Vorschulkindern und Erstklässlern. *Lernen und Lernstörungen, 8*(3), 155–165. Retrived from https://doi.org/10.1024/2235-0977/a000216.

Wille, A. (2020). Mathematische Gebärden der Österreichischen Gebärdensprache aus semiotischer Sichtt. In G. Kadunz, *Zeichen und Sprache im Mathematikunterricht,* 193–214. Springer Spektrum. Retrieved from https://doi.org/10.1007/978-3-662-61194-4_9.

Wille, A. (2019). Gebärdensprachliche Videos für Textaufgaben im Mathematikunterricht: Barrieren abbauen und Stärken gehörloser Schülerinnen und Schüler nutzen. *Mathematik differenziert, 2019*(3), 38-44. https://www.westermann.de/anlage/4617337/Gebaerdensprachliche-Videos-fuer-Textaufgaben-im-Mathematikunterricht-Barrieren-abbauen-und-Staerken-gehoerloser-Schuelerinnen-und-Schueler-nutzen.

Wille, A. (2019, 3-8. March). Einsatz von Materialien zur Bruchrechnung für gehörlose Schülerinnen und Schüler im inklusiven Mathematikunterricht. *53. Jahrestagung der Gesellschaft für Didaktik der Mathematik (GDM).* Regensburg, Germany.

Wille, A. (2018, 5.-9. March). Materialien für den Mathematikunterricht gehörloser Schülerinnen und Schüler. *52. Jahrestagung der Gesellschaft für Didaktik der Mathematik und der Deutschen Mathematiker-Vereinigung (GDMV).* Padeborn, Germany.

Wille, A. M. (2008). Aspects of the concept of a variable in imaginary dialogues written by students. In O. Figueras, J. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education (PME32).* Vol. 4. (pp. 417–424). Cinvestav-UMSNH, Mexico: PME.

Winkenbach, B. (2011). Die Käferwanderung auf dem Würfel. *Stochastik in der Schule,* 31, 23–27.

Wittmann, E. Ch. (2001). Grundkonzeption des Zahlenbuchs. In Wittmann et al. (Eds.), *Das Zahlenbuch: Mathematik im 1. Schuljahr: Lehrerband.* Klett. https://www.mathe2000.de/sites/default/files/das-zahlenbuch-grundkonzeption.pdf.

Yashkova, N. (1988). *Visual thinking of deaf children.* Pedagogika.

Zajceva, G. L. (2000) Žestowaja reč'. VLADOC.

Zarfaty, Y., Nunes, T., & Bryant, P.E. (2004). The performance of young deaf children in spatial and temporal number tasks. Journal of Deaf Studies and Deaf Education, 9(3), 315-26, https://doi.org/10.1093/deafed/enh034.